Integer points close to a transcendental curve and correctly-rounded evaluation of a function

Nicolas Brisebarre (C.N.R.S.) and Guillaume Hanrot (É.N.S. Lyon)

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(Binary) Floating Point (FP) Arithmetic

Given

$$\begin{cases} \text{a precision} & p \geqslant 1, \\ \text{a set of exponents} & E_{\mathsf{min}}, \cdots, E_{\mathsf{max}}. \end{cases}$$

A finite FP number x is represented by 2 integers:

- integer significand M, $2^{p-1} \leqslant |M| \leqslant 2^p - 1$,
- exponent E, $E_{\mathsf{min}} \leqslant E \leqslant E_{\mathsf{max}}$

such that

$$x = \frac{M}{2^{p-1}} \times 2^E.$$

IEEE Precisions

IEEE 754 standard (1984 then 2008).

See http://en.wikipedia.org/wiki/IEEE_floating_point

	precision p	min. exponent E_{min}	maximal exponent E_{max}
binary32 (single)	24	-126	127
binary64 (double)	53	-1022	1023
binary128 (quadruple)	113	-16382	16383

We have $x = \frac{M}{2^{p-1}} \times 2^E$ with $2^{p-1} \leqslant |M| \leqslant 2^p - 1$

and $E_{\text{min}} \leqslant E \leqslant E_{\text{max}}$.

Rounding modes

In the IEEE 754 standard, the user defines an *active rounding mode*.

In this talk, we use:

- round to nearest (default). If $x \in \mathbb{R}$, $\mathrm{RN}(x)$ is the floating-point number that is the closest to x. In case of a tie, value whose integral significand is even.

A breakpoint is a point where the rounding function changes.

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Here, breakpoint $=$ the middle of two consecutive FP numbers.

Correct rounding

A correctly-rounded operation whose entries are FP numbers must return what we would get by infinitely precise operation followed by rounding.

It brings various benefits. In particular:

- accuracy and portability are improved;
- FP arithmetic becomes a structure in itself, that can be studied.

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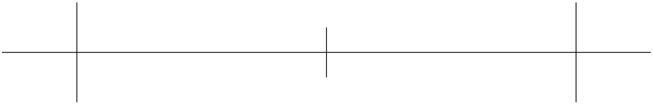
- FP arithmetic becomes a structure in itself, that can be studied.

IEEE-754 (1985): Correct rounding for $+$, $-$, $\times$, $\div$, $\sqrt{}$ and some conversions.

IEEE-754 (2008): suggests correct rounding for some elementary functions ($\sqrt[n]{}$, $\sin$, $\cos$, $\arcsin$, $\arccos$, $\tan$, $\arctan$, $\exp$, $\log$, $\sinh$, $\cosh$...).

The Table Maker's Dilemma

$x \in [1,2),\ x = \dfrac{j}{2^{p-1}},\ j \in \mathbb{Z},\ 2^{p-1} \leqslant j \leqslant 2^p - 1,\ f(x) \in [1,2)$



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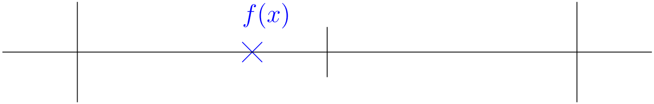

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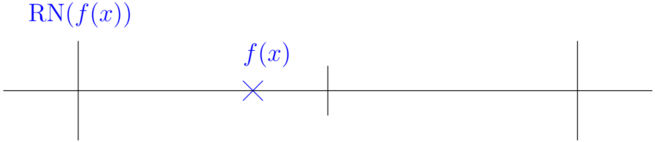

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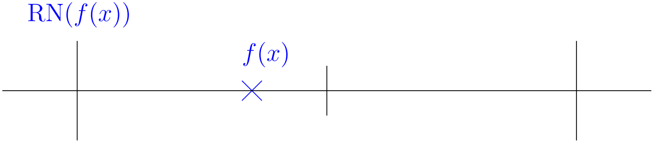

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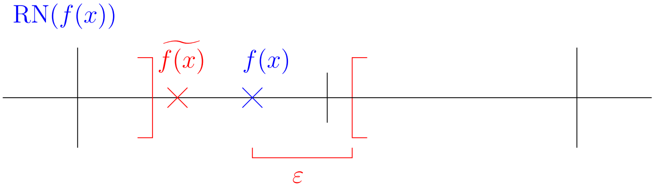

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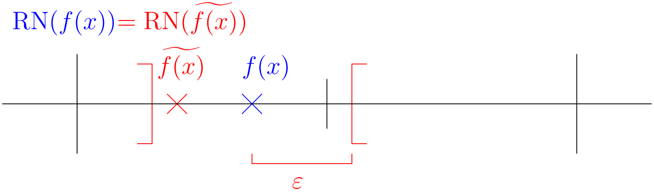

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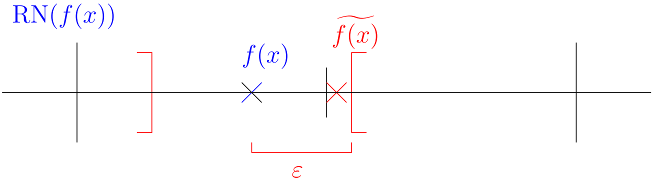

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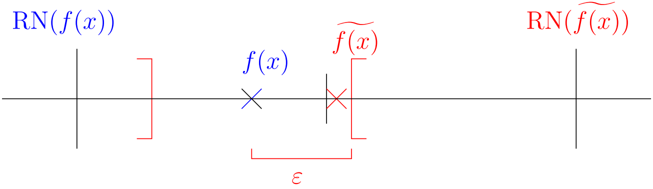

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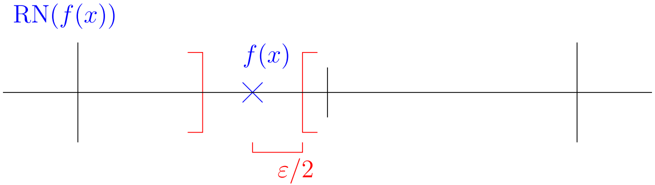

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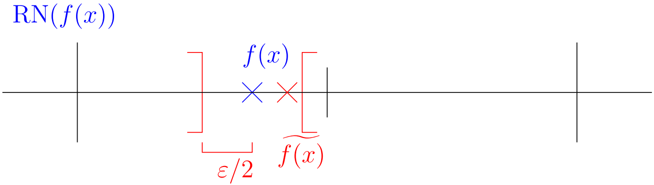

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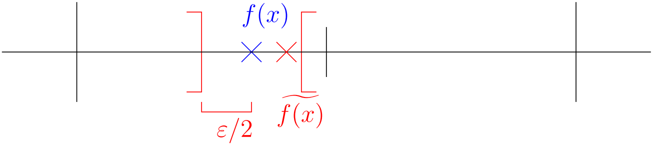
$\mathrm{RN}(f(x)) = \mathrm{RN}(\widetilde{f(x)})$



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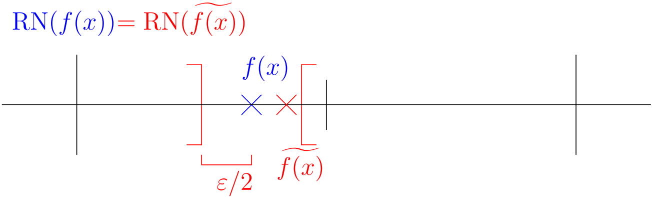

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Potential issues:

- What if $f(x)$ is a breakpoint?
- What about the number of subdivisions?
- ε should be uniform!

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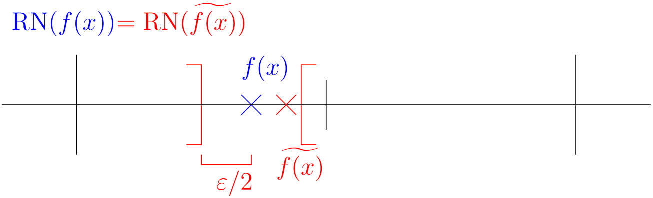

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- ε should be uniform! And as large as possible!

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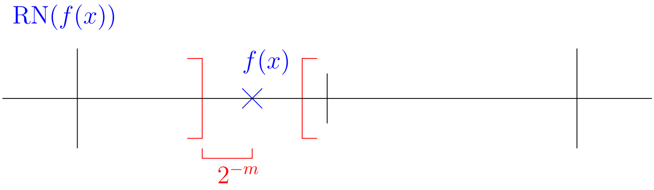

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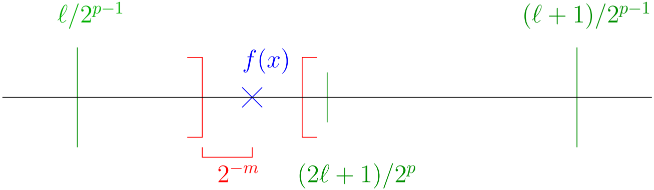

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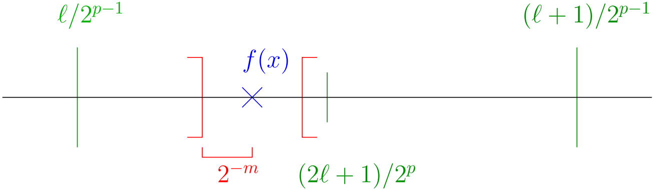

We want to find $m \in \mathbb{N}$ s.t.

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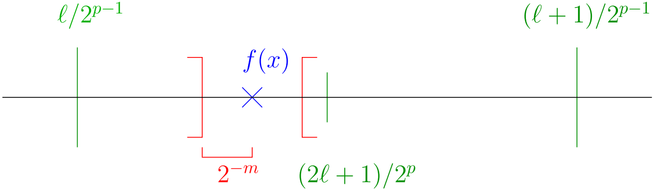

We want to find $m \in \mathbb{N}$, as small as possible, s.t. for all FP x:

- either there exists $\ell \in [\![2^{p-1}, 2^p - 1]\!]$ s.t. $f(x) = (2\ell + 1)/2^p$,
- or

$$\text{for all } k \in [\![2^{p-1}, 2^p - 1]\!], \left| f(x) - \frac{2k+1}{2^p} \right| \geqslant 2^{-m}.$$

The Table Maker's Dilemma: Diophantine Problems

Assume, w.l.o.g., that x and $f(x) \in [1, 2)$.

Q. (TMD) We want to determine $m \in \mathbb{N}$, as small as possible, s.t. for all $j \in [\![2^{p-1}, 2^p - 1]\!]$,

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Some insight (Warning: Hand-waving!)...

Assume there exist $x \in [1,2)$, $k \in \mathbb{N} \setminus \{0\}$ and $\ell \in [\![2^{p-1}, 2^p - 1]\!]$ s.t.

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The infinitely precise significand y of $f(x)$ has the form:

$$y = y_0.y_1y_2 \cdots y_{p-1} \underbrace{01111111 \cdots 11}_{k \text{ bits}} xxxxx \cdots$$

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Assessing the Heuristic: the Example of $\sin$

Here, $f = \sin$ over $[1, 2)$, $p = 16$.

k	Actual number of occurrences	Expected number of occurrences
1	16397	16384
2	8151	8192
3	4191	4096
4	2043	2048
5	1010	1024
6	463	512
7	255	256

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Here, $f = \sin$ over $[1, 2)$, $p = 16$.

k	Actual number of occurrences	Expected number of occurrences
8	131	128
9	62	64
10	35	32
11	16	16
12	7	8
13	6	4
14	0	2
15	1	1

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Here, the heuristic seems reasonable.

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→ roughly,

$$\text{"}m_{opt} \sim 2p\text{"} \qquad \text{(Q).}$$

The Table Maker's Dilemma: an Example

Consider the function 2^x and the binary64 FP number (base $2, p = 53$)

$$x = \frac{8520761231538509}{2^{62}}$$

We have

$$\begin{aligned} 2^{52+x} &= 4509371038706515.4999999999999999994026\cdots(\text{decimal}) \\ &= \underbrace{1\cdots}_{53 \text{ bits}}.0\underbrace{1\cdots\cdots\cdots\cdots1}_{60 \text{ consecutive } 1's}0\cdots(\text{binary}). \end{aligned}$$

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Hardest-to-round (HR) case for function 2^x and binary64 FP numbers.
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Function f: $\sqrt[n]{\ }$, $\sin$, $\cos$, $\arcsin$, $\arccos$, $\tan$, $\arctan$, $\exp$, $\log$, $\sinh$, $\cosh$...

Proving the Heuristic

NB, G. Hanrot and O. Robert (2017)

Let $f : [1, 2) \mapsto [1, 2)$, $f \in \mathcal{C}^2$, let $k \in \mathbb{N}$.

Determine the proportion of $j \in [\![2^{p-1}, 2^p - 1]\!]$ s.t. there exists $\ell \in [\![2^{p-1}, 2^p - 1]\!]$ with

$$\left| f\left(\frac{j}{2^{p-1}}\right) - \frac{2\ell + 1}{2^p}\right| < \frac{1}{2^{p+k}}.$$

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Proposition

For $\exp$ *over* $[1,2)$*, if* $p \geqslant 24$*, the heuristic is valid for* $0 \leqslant k < p/3$*.*

The Table Maker's Dilemma: Diophantine Problems

Assume, w.l.o.g., that x and $f(x) \in [1, 2)$.

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The Table Maker's Dilemma: First Challenge

A breakpoint is a point where the rounding function changes. In this talk, it is the middle of two consecutive FP numbers.

First challenge:

- Determine the set BP_f of all the FP numbers $x \in [1,2)$ such that $f(x)$ is a breakpoint.

In other words, determine all $j, \ell \in [\![2^{p-1}, 2^p - 1]\!]$ s.t.

$$f\left(\frac{j}{2^{p-1}}\right) = \frac{2\ell + 1}{2^p}.$$

State of the Art: Theoretical Results

Determine all $j, \ell \in [\![2^{p-1}, 2^p - 1]\!]$ s.t.

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In other words, if Γ_g graph of g, determine $(2^{p-1}\Gamma_g) \cap \mathbb{Z}^2$.

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$$|N\Gamma_g \cap \mathbb{Z}^2| \leqslant \frac{3}{(4\pi)^{1/3}} N^{2/3} + O(N^{1/3}), \text{ for any } N \in \mathbb{N} \setminus \{0\}.$$

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The bound is sharp for some $g \in \mathcal{C}^2$.

Let $N = 2^{p-1}$: we have $2^{2(p-1)/3}$ instead of $O(1)$.

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Theorem (Bombieri & Pila, 1989)

Let $g : [0, 1] \mapsto [0, 1]$ *be a transcendental analytic function, then, for any* $\varepsilon > 0$*, there exists* $c(g, \varepsilon) > 0$ *s.t., for any* $N \in \mathbb{N} \setminus \{0\}$*,*

$$|N\Gamma_g \cap \mathbb{Z}^2| \leqslant c(g, \varepsilon) N^{\varepsilon}.$$

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Transcendental elementary Functions $\sin$, $\cos$, $\arcsin$, $\arccos$, $\tan$, $\arctan$, $\exp$, $\log$, $\sinh$, $\cosh$. Hermite-Lindemann's theorem: $\alpha \neq 0$ algebraic $\Rightarrow$ e^{α} transcendental.

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What about the Euler Gamma function? For $\mathrm{Re}(z) > 0$,

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} \mathrm{d}t.$$

Very little is known:
$\Gamma(1/2), \Gamma(1/3), \Gamma(1/4), \Gamma(1/6), \Gamma(2/3), \Gamma(3/4), \Gamma(5/6)$ are transcendental and there are some partial irrationality results.

Our setting

Let $f : [1, 2) \mapsto [1, 2)$, f is transcendental and analytic in the neighborhood of $[1, 2)$.

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Let $g \in \mathcal{C}([a,b])$,

$$\|g\|_{\infty,[a,b]} := \max_{x\in[a,b]} |g(x)|.$$

Our Approach: Polynomial Interpolation and Lattice Basis Reduction

We want to find all $2^{p-1} \leqslant i, j \leqslant 2^p - 1$ s.t.

$$f\left(\frac{i}{2^{p-1}}\right) = \frac{2j+1}{2^p},$$

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We build a trap! We find a partition of $[1,2) = \cup_\ell I_\ell$ such that, for all ℓ, we can compute $P_{\ell,1}, P_{\ell,2} \in \mathbb{Z}[X,Y] \setminus \{0\}$ with

$$\left|P_{\ell,k}(2^{p-1}u, 2^p f(u))\right| < 1, k = 1, 2,$$

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We eliminate (heuristic assumption!) one of the two variables and we get i and j, if they exist (Coppersmith; Boneh & Durfee; Stehlé).

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Actually, the lattice reduction step makes it possible to refine the choice of the basis.

Basis in Use

Let $d \in \mathbb{N}$, if $X = 2^{p-1}x$ and $Y = 2^p f(x)$ the elements of the basis that we use are:

$$
\begin{array}{cccccc}
1, & & & & & \\
X, & Y, & & & & \\
X^2, & XY, & Y^2, & & & \\
\vdots & \vdots & \vdots & \ddots & & \\
X^{d-1}, & X^{d-2}Y, & X^{d-3}Y^2, & \cdots & Y^{d-1}, & \\
X^d, & X^{d-1}Y, & X^{d-2}Y^2, & \cdots & XY^{d-1}, & Y^d,
\end{array}
$$

i.e., the basis of use is $\left((2^{p-1}x)^k(2^p f(x))^\ell\right)_{\substack{0\leqslant \ell \leqslant d \\ 0 \leqslant k \leqslant d-\ell}}$.

Dimension $N = (d+1)(d+2)/2$.

Ensuring the Smallness of a Function: Interpolation at Chebyshev Nodes

Definition

Let $n \in \mathbb{N}$, the Chebyshev nodes of the first kind of order n are the points $\mu_k = \cos\left(\frac{(2k+1)\pi}{2(n+1)}\right), k = 0, \ldots, n$.

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The polynomial p_n is a quasi-optimal uniform approximation to φ:

$$\|\varphi - p_n\|_{\infty,[-1,1]} \leqslant 2\left(\frac{1}{\pi}\log(n+1)+1\right)\min_{Q\in\mathbb{R}_n[x]}\|\varphi - Q\|_{\infty,[-1,1]}.$$

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Let $P \in \mathbb{R}_n[X]$, we have

$$\max_{x\in[-1,1]} |P(x)| \leqslant \left(\frac{2}{\pi}\log(n+1) + 1\right) \max_{k=0,\ldots,n} |P(\mu_k)|.$$

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Bounding the Remainder - Bernstein Ellipse

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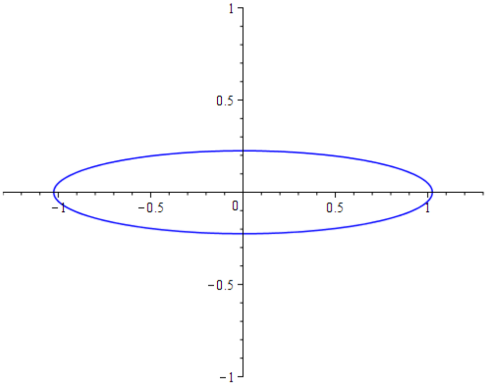

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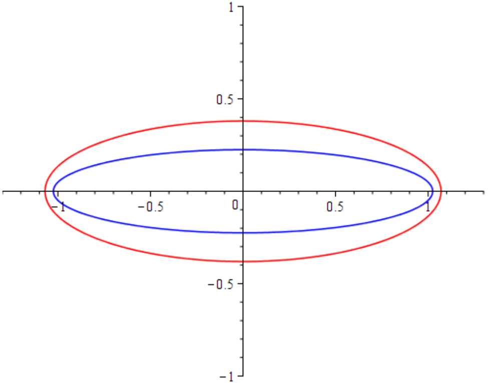

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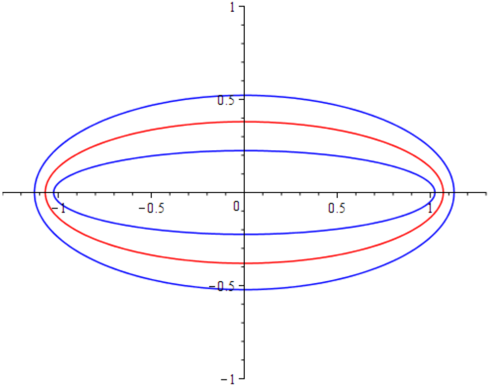

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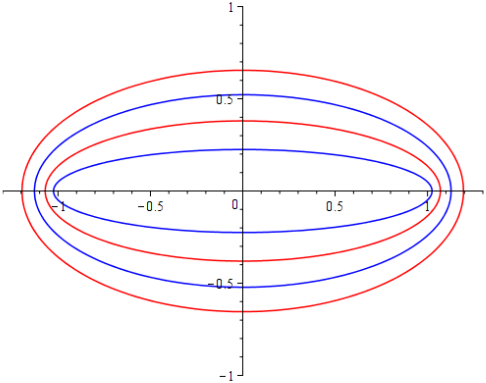

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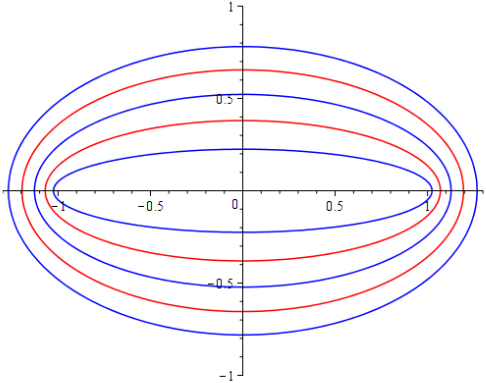

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$$\|\varphi - p_n\|_{\infty,[-1,1]} \leqslant \frac{4M_\rho(\varphi)}{\rho^n(\rho - 1)}.$$

where $M_\rho(\varphi) = \max_{z\in\mathcal{E}_\rho} |\varphi(z)|$.

Interpolation at Chebyshev Nodes and Uniform Approximation

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Interpolation at Chebyshev Nodes and Uniform Approximation: The case of $[a, b]$

Let $I = [a, b]$, one defines

- scaled Chebyshev nodes of the first kind of order n:
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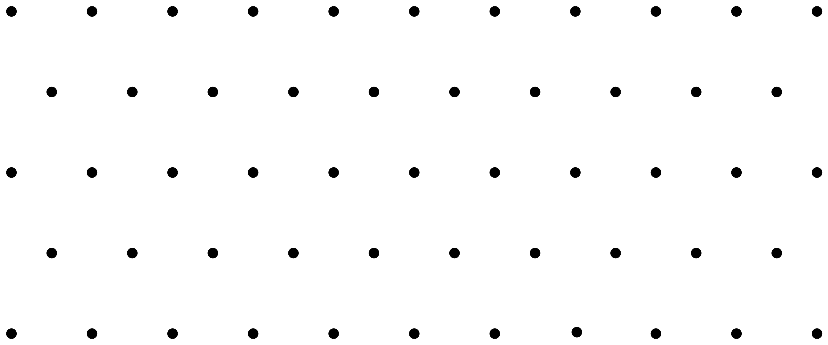
Lattice Basis Reduction



An Approach based on Lattice Basis Reduction

Definition

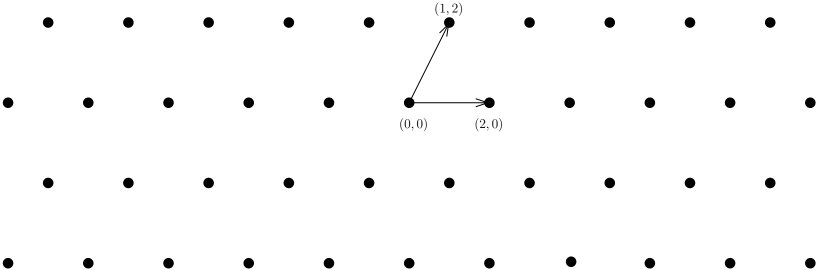
Let L be a nonempty subset of $\mathbb{R}^d$, L is a lattice iff there exists a set of vectors $b_1, \dots, b_k$ $\mathbb{R}$-linearly independent such that

$$L = \mathbb{Z}.b_1 \oplus \cdots \oplus \mathbb{Z}.b_k.$$

$(b_1, \dots, b_k)$ is a basis of the lattice L.

Examples. $\mathbb{Z}^d$, every subgroup of $\mathbb{Z}^d$.

Example: The Lattice $\mathbb{Z}(2,0) \oplus \mathbb{Z}(1,2)$



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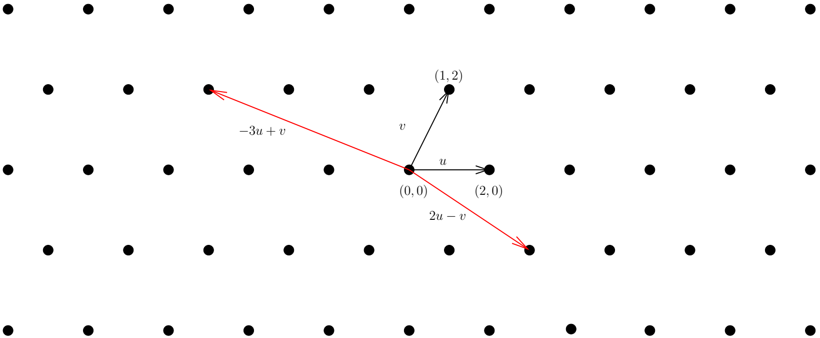

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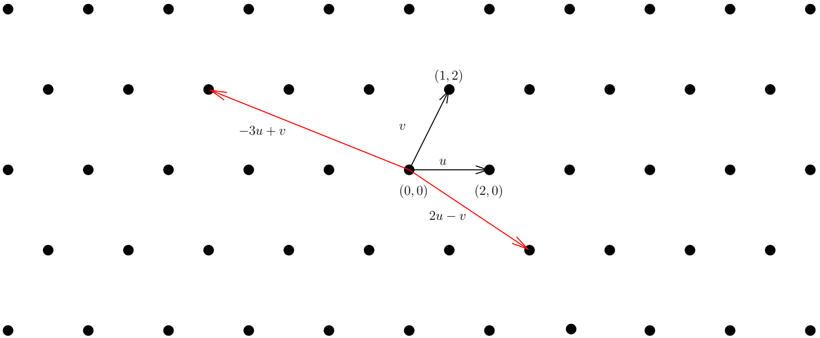

SVP (Shortest Vector Problem)

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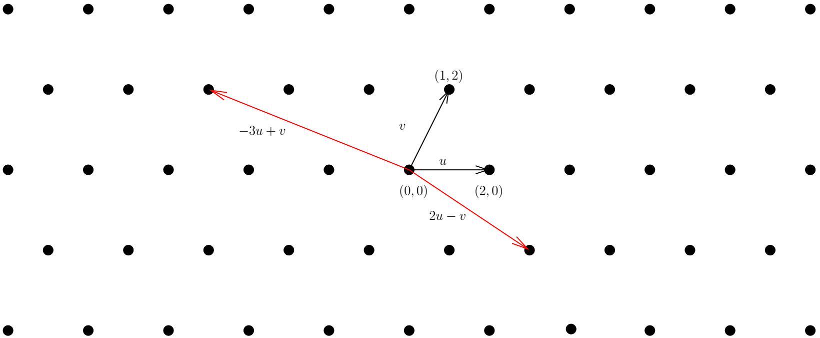

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Lenstra-Lenstra-Lovász Algorithm

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Factoring Polynomials with Rational Coefficients, A. K. Lenstra, H. W. Lenstra and L. Lovász, Math. Annalen **261**, 515-534, 1982.

The LLL algorithm gives an approximate solution to SVP in polynomial time.

Lenstra-Lenstra-Lovász Algorithm

Theorem

Let L *a lattice of dimension* k*.*
LLL provides a basis $(b_1, \ldots, b_k)$ *made of "pretty" short vectors. We have* $||b_1|| \leqslant 2^{(k-1)/2}\lambda_1(L)$ *where* $\lambda_1(L)$ *denotes the norm of a shortest nonzero vector of* L*.*
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Let $(b_1, \ldots, b_k)$ be an LLL-reduced basis L then

$$||b_1|| \leqslant 2^{k/2}(\text{vol } L)^{1/k} \quad \text{and} \quad ||b_2|| \leqslant 2^{k/2}(\text{vol } L)^{\frac{1}{k-1}}.$$

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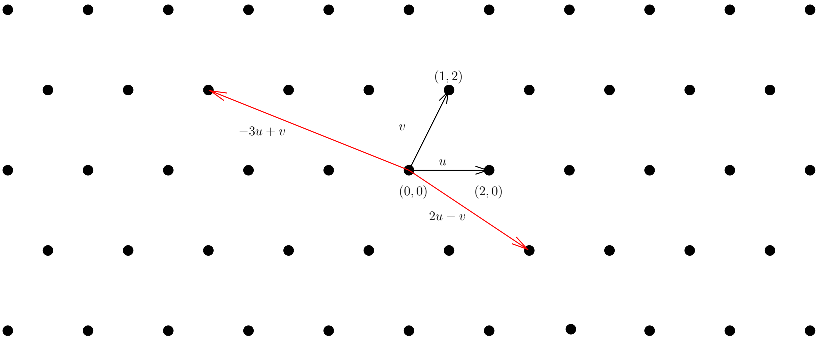

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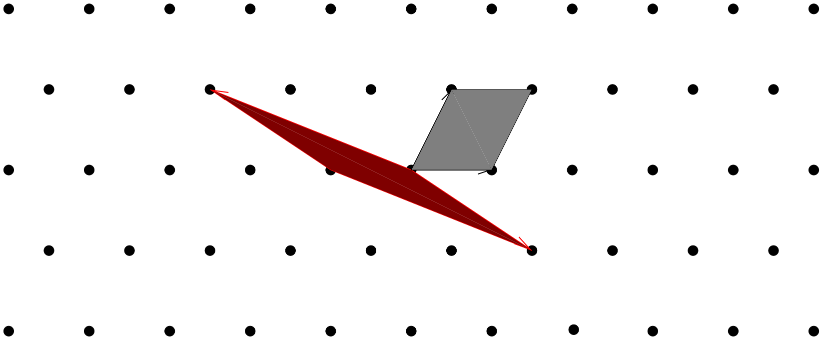

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Let $d \in \mathbb{N}$, $P_1 = \sum_{0 \leqslant u+v \leqslant d} \alpha_{u,v} X^u Y^v$ and $P_2 = \sum_{0 \leqslant u+v \leqslant d} \beta_{u,v} X^u Y^v \in \mathbb{Z}[X, Y]$. We want to have

$$\left|P_k(2^{p-1}x, 2^p f(x))\right| < 1, \qquad k = 1, 2,$$

for all $x \in I = [a, b]$.

How do we compute P_1 and P_2? The Lattice

Let $d \in \mathbb{N} \setminus \{0\}$ and $N = (d+1)(d+2)/2$. Let $(x_j)_{0 \leqslant j \leqslant N-1}$ denote Chebyshev nodes for the interval $I = [a, b]$.
We introduce, for $0 \leqslant k \leqslant d, 0 \leqslant \ell \leqslant d-k$,

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$$\begin{pmatrix}
f_{0,0}(x_0) & \cdots & f_{0,0}(x_{N-1}) & r_{0,0} & 0 & \cdots & \cdots & 0 \\
f_{0,1}(x_0) & \cdots & f_{0,1}(x_{N-1}) & 0 & r_{0,1} & 0 & \cdots & 0 \\
\vdots & \cdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
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Hence, the function g is "small" !

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If the vector V is small, the function g is "small" !

LLL gives us two such short vectors, as long as the volume of the lattice is small.

How do we compute P_1 and P_2? Volume of the Lattice

Let $d \in \mathbb{N} \setminus \{0\}$ and $N = (d+1)(d+2)/2$. Let $(x_j)_{0 \leqslant j \leqslant N-1}$ denote Chebyshev nodes for the interval $I = [a, b]$.
We introduce, for $0 \leqslant k \leqslant d, 0 \leqslant \ell \leqslant d-k$,

$$f_{k,\ell}(x) = (2^{p-1}x)^{\ell}(2^p f(x))^k \text{ and } r_{k,\ell} = \frac{4M_\rho(f_{k,\ell})}{\rho^{N-1}(\rho - 1)}.$$

Our lattice: $\mathcal{L}$ generated by the rows of

$$\begin{array}{c} f_{0,0} \\ f_{0,1} \\ \vdots \\ \\ f_{d-1,1} \\ f_{d,0} \end{array}
\begin{pmatrix}
f_{0,0}(x_0) & \cdots & f_{0,0}(x_{N-1}) & r_{0,0} & 0 & \cdots & \cdots & 0 \\
f_{0,1}(x_0) & \cdots & f_{0,1}(x_{N-1}) & 0 & r_{0,1} & 0 & \cdots & 0 \\
\vdots & \cdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
 & & & \vdots & & & & \\
f_{d-1,1}(x_0) & \cdots & f_{d-1,1}(x_{N-1}) & \vdots & \cdots & 0 & r_{d-1,1} & 0 \\
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Its volume $\mathrm{vol}(L)$: determinant of the matrix.

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Let $d \in \mathbb{N} \setminus \{0\}$ and $N = (d+1)(d+2)/2$. We have, for $\rho > 1$,

$$\mathrm{vol}(\mathcal{L})^{1/N} \leqslant O(N) \frac{2^{2pd/3}}{\rho^{(N-1)/2}} \left| \frac{b-a}{2}\rho + \frac{b+a}{2} \right|^{d/3} M_{\rho,a,b}(f)^{d/3}.$$

where $M_{\rho,a,b}(f) = \max_{z \in \mathcal{E}_{\rho,a,b}} |f(z)|$ and
$\mathcal{E}_{\rho,a,b} = \left\{ \frac{b-a}{2} \frac{\rho e^{i\theta} + \rho^{-1} e^{-i\theta}}{2} + \frac{a+b}{2}, \theta \in [0, 2\pi] \right\}$.

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If $[a, b] = [1, 2]$, 40 CPU minutes for $p = 53$ and 46 CPU days for $p = 113$.

Computations

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Our experiments were done in Sagemath[1] and heavily use the Arb[2] and fplll[3] libraries

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The Table Maker's Dilemma

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Holy Grail: $m \sim 2p$. True for $p = 53$ (V. Lefèvre et al).

Our results

Thanks to an extension of the presented ideas, we obtain for instance, for $p = 113$, for all $j, \ell \in [\![2^{p-1}, 2^p - 1]\!]$ and

$$\left| \exp\left(\frac{j}{2^{p-1}}\right) - \frac{2\ell + 1}{2^p} \right| \geqslant \frac{1}{2^{12p}}$$

in less than 9 days.

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Still, this work should hopefully help paving the way for correctly rounded elementary functions in IEEE binary128/quadruple precision.