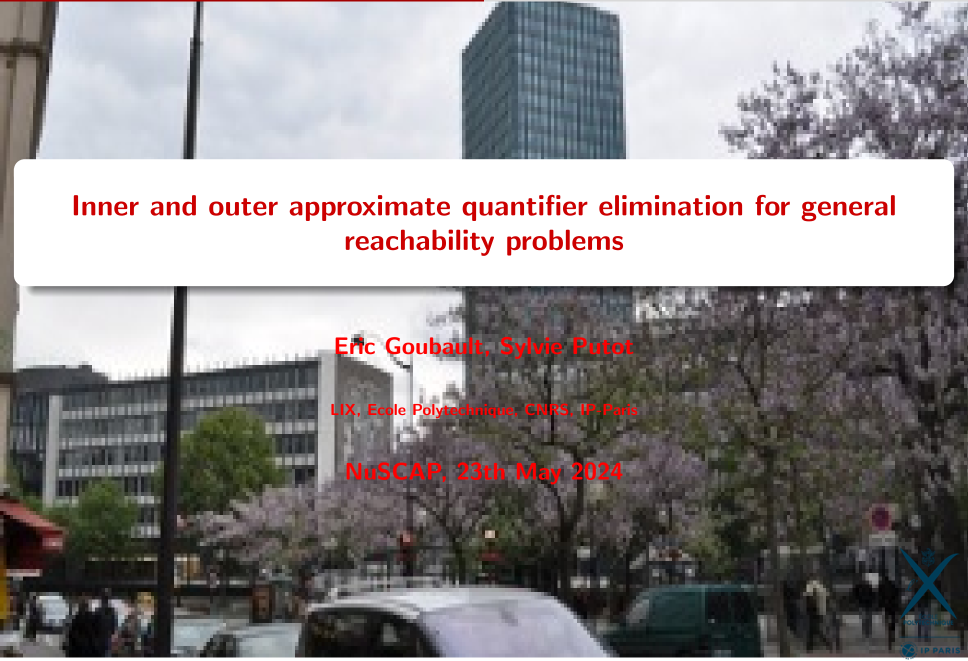


# Inner and outer approximate quantifier elimination for general reachability problems

Eric Goubault, Sylvie Putot

LIX, Ecole Polytechnique, CNRS, IP-Paris

NuSCAP, 23th May 2024

# Quantified reachability problems?

## Sets of reachable states

For discrete or continuous systems $\varphi$ (control $u$, perturbations $w$, initial states $x_0$)


## Reachability under uncertainties

Maximal reachability: set of states $z$ (maximally) reachable at time $s$:

$$\{z \mid \exists x_0 \in \boldsymbol{Z}_0,\ \exists u : [0,s] \to \mathbb{U},\ \exists w \in [0,s] \to \mathbb{W} \text{ s.t. } \varphi(s; x_0, u, w) = z\}$$

# Quantified reachability problems?

## Sets of reachable states

For discrete or continuous systems $\varphi$ (control $u$, perturbations $w$, initial states $x_0$), intractable in general: need for outer-approximations


## Reachability under uncertainties

Maximal reachability: set of states $z$ (maximally) reachable at time $s$:

$$\{z \mid \exists x_0 \in \boldsymbol{Z}_0, \exists u : [0,s] \to \mathbb{U}, \exists w \in [0,s] \to \mathbb{W} \text{ s.t. } \varphi(s; x_0, u, w) = z\}$$

# Quantified reachability problems?

## Sets of reachable states

For discrete or continuous systems $\varphi$ (control $u$, perturbations $w$, initial states $x_0$) , intractable in general: need for inner and outer-approximations


## Reachability under uncertainties

Maximal reachability: set of states $z$ (maximally) reachable at time $s$:

$$\{z \mid \ \exists x_0 \in \boldsymbol{Z}_0, \exists u : [0,s] \to \mathbb{U}, \exists w \in [0,s] \to \mathbb{W} \text{ s.t. } \varphi(s; x_0, u, w) = z \ \}$$

# Quantified reachability problems?

## Sets of reachable states

For discrete or continuous systems $\varphi$ (control $u$, perturbations $w$, initial states $x_0$) , intractable in general: need for inner and outer-approximations


## Reachability under uncertainties

Minimal reachability: set of states $z$ (minimally) reachable at time $s$:

$$\{z \mid \ \forall u : [0,s] \to \mathbb{U},\ \forall w \in [0,s] \to \mathbb{W},\ \exists x_0 \in \boldsymbol{Z}_0 \text{ s.t. } \varphi(s; x_0, u, w) = z\ \}$$

# Quantified reachability problems?

## Sets of reachable states

For discrete or continuous systems $\varphi$ (control $u$, perturbations $w$, initial states $x_0$) , intractable in general: need for inner and outer-approximations


## Reachability under uncertainties

Or "robust reachability" (E. Goubault, S. Putot: Inner and outer reachability for the verification of control systems. HSCC 2019):

$$\{z \mid \forall w \in [0,s] \to \mathbb{W},\ \exists x_0 \in \mathbb{X}_0,\ \exists u : [0,s] \to \mathbb{U}, \varphi(s; x_0, u, w) = z\}$$

# Quantified reachability problems?

## Classical reachability (inner and outer approximations)


"Robust reachability" (HSCC 2019):
$\{z \mid \forall w \in [0,s] \to \mathbb{W},\ \exists x_0 \in \mathbb{X}_0, \exists u : [0,s] \to \mathbb{U}, z = \varphi(t; x_0, u, w)\}$

## This presentation: add more quantifiers!

Why? Wait for next slide!

## This presentation is part of a larger programme

- Fast and precise set-based methods for guaranteed inner/outer approximations of Quantifier Elimination (QE, or Quantified Constraint Solving)
- We focus here on simple 0th-order (interval based) approximations of QE, useful for "general reachability" problems

# More, or different alternations of quantifiers?

### Reminder: robust reachability of HSCC 2019

Given $\varphi(t; x_0, u, w)$ the flow of an ODE at time $t$ from $x_0$ with control $u$ and disturbance $w$, for time $t \in [0, T]$, compute:

$$R_{\forall\exists}(\varphi)(t) = \{z \mid \forall w \in [0,s] \to \mathbb{W},\ \exists x_0 \in \mathbb{X}_0,\ \exists u \in [0,s] \to \mathbb{U}, z = \varphi(t; x_0, u, w)\}$$

(can a controller compensate disturbances or change of values of parameters that are known to the controller?)

### Alternative problem (control is not aware of perturbations)

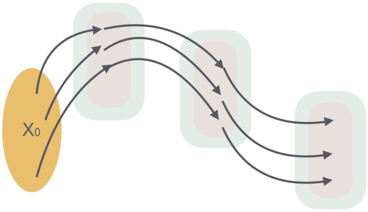
Can a controller not knowing the disturbance still reach the target, up to some (time) relaxation?

$$R_{\exists\forall\exists}(\varphi) = \{z \in \mathbb{R}^m \mid \exists u \in [0,s] \to \mathbb{U},\ \exists x_0 \in \mathbb{X}_0,\ \forall w \in [0,s] \to \mathbb{W},\ \exists s \in [0,T],\ z = \varphi(s; x_0, u, w)\}$$

# But also

## Motion planning

- Find possible waypoints and final state, for a controller that takes $k$ constrained actions
- Gives $k$ alternations of $\forall\exists$ quantifiers, for $k$ waypoints




## General temporal logics formulas, and hyperproperties

- behavioral robustness,
- comparisons of controllers

Etc.

# Problem statement

## General quantified problems

For $f : \mathbb{R}^p \to \mathbb{R}^m$ (e.g. flow function etc.), generally supposed continuously differentiable, consider alternations of quantifiers $\forall/\exists$ reachability problem:

$$R_{\boldsymbol{p}}(f) = \{z \in \mathbb{R}^m \mid Q_1 x_1 \in [-1,1],\ Q_2 x_2 \in [-1,1],\ \ldots,\\ Q_{p-1} x_{p-1} \in [-1,1], Q_p x_p \in [-1,1],\ z = f(x_1, x_2, \ldots, x_p)\}$$

where $Q_i = \forall$ or $Q_i = \exists$.

## Discussed in the paper

E. Goubault, S. Putot: Inner and outer approximate quantifier elimination for general reachability problems. HSCC 2024

- Up to reparametrization, quantified problems with other boxes than $[-1,1]^{j_i}$
- Also possible to consider more general sets over which to quantify variables $x_i$ by suitable outer and inner approximations as boxes
- Can consider e.g. control $u$ and disturbance $w$ as piecewise constant signals over a bounded time horizon.

# Steps of the construction

Step 1: The case of linear scalar functions $f: \mathbb{R}^p \rightarrow \mathbb{R}$

Exact solution via a basic two-player game

# Steps of the construction

Step 1: The case of linear scalar functions $f: \mathbb{R}^p \to \mathbb{R}$

Exact solution via a basic two-player game

Step 2: The case on non-linear scalar functions $f: \mathbb{R}^p \to \mathbb{R}$

Use suitable inner and outer approximate linearizations

# Steps of the construction

Step 1: The case of linear scalar functions $f: \mathbb{R}^p \to \mathbb{R}$

Exact solution via a basic two-player game

Step 2: The case on non-linear scalar functions $f: \mathbb{R}^p \to \mathbb{R}$

Use suitable inner and outer approximate linearizations

Step 3: The general case, non-linear functions $f: \mathbb{R}^p \to \mathbb{R}^n$

Use relaxations of quantified formulas involved at different components of $f$

# Step 1, quantified reachability for scalar linear functions

# Let us play a simple two-player game!

$f$ is the affine function $f(x_1, x_2, \ldots, x_p) = \delta_0 + \delta_1 x_1 + \delta_2 x_2 + \ldots + \delta_p x_p$; variables $x_i$ are either quantified by $Q_i$ being $\exists$ or by $\forall$.

## The players

($\exists$-player)

($\forall$-player)

# Let us play a simple two-player game!

$f$ is the affine function $f(x_1, x_2, \ldots, x_p) = \delta_0 + \delta_1 x_1 + \delta_2 x_2 + \ldots + \delta_p x_p$; variables $x_i$ are either quantified by $Q_i$ being $\exists$ or by $\forall$.

## The rules of the game

Compute $S = [\underline{S}, \overline{S}]$, the quantified reachable interval; initially $S = \{f(0, \ldots, 0)\}$

# Let us play a simple two-player game!

$f$ is the affine function $f(x_1, x_2, \ldots, x_p) = \delta_0 + \delta_1 x_1 + \delta_2 x_2 + \ldots + \delta_p x_p$; variables $x_i$ are either quantified by $Q_i$ being $\exists$ or by $\forall$.

## The rules of the game

Compute $S = [\underline{S}, \overline{S}]$, the quantified reachable interval; initially $S = \{f(0, \ldots, 0)\}$

- At round $i$ from $p$ to 1,  plays if $Q_i = \exists$,  plays if $Q_i = \forall$

# Let us play a simple two-player game!

$f$ is the affine function $f(x_1, x_2, \ldots, x_p) = \delta_0 + \delta_1 x_1 + \delta_2 x_2 + \ldots + \delta_p x_p$; variables $x_i$ are either quantified by $Q_i$ being $\exists$ or by $\forall$.

## The rules of the game

Compute $S = [\underline{S}, \overline{S}]$, the quantified reachable interval; initially $S = \{f(0, \ldots, 0)\}$

- At round $i$ from $p$ to 1,  plays if $Q_i = \exists$,  plays if $Q_i = \forall$
    -  widens $S$ by $[-\delta_i, \delta_i]$ ($\underline{S}$-$=\delta_i$, , $\overline{S}$+$=\delta_i$)

# Let us play a simple two-player game!

$f$ is the affine function $f(x_1, x_2, \ldots, x_p) = \delta_0 + \delta_1 x_1 + \delta_2 x_2 + \ldots + \delta_p x_p$; variables $x_i$ are either quantified by $Q_i$ being $\exists$ or by $\forall$.

## The rules of the game

Compute $S = [\underline{S}, \overline{S}]$, the quantified reachable interval; initially $S = \{f(0, \ldots, 0)\}$

- At round $i$ from $p$ to 1,  plays if $Q_i = \exists$,  plays if $Q_i = \forall$
  -  widens $S$ by $[-\delta_i, \delta_i]$ $(\underline{S}$-$=\delta_i$, , $\overline{S}+=\delta_i)$
  -  shrinks $S$ by $[-\delta_i, \delta_i]$ $(\underline{S}+=\delta_i$, , $\overline{S}-=\delta_i)$

# Let us play a simple two-player game!

$f$ is the affine function $f(x_1, x_2, \ldots, x_p) = \delta_0 + \delta_1 x_1 + \delta_2 x_2 + \ldots + \delta_p x_p$; variables $x_i$ are either quantified by $Q_i$ being $\exists$ or by $\forall$.

## The rules of the game

Compute $S = [\underline{S}, \overline{S}]$, the quantified reachable interval; initially $S = \{f(0, \ldots, 0)\}$

- At round $i$ from $p$ to 1,  plays if $Q_i = \exists$,  plays if $Q_i = \forall$
  -  widens $S$ by $[-\delta_i, \delta_i]$ ($\underline{S}$-=$\delta_i$, , $\overline{S}$+=$\delta_i$)
  -  shrinks $S$ by $[-\delta_i, \delta_i]$ ($\underline{S}$+=$\delta_i$, , $\overline{S}$−=$\delta_i$)
- Stops either after step $i = 1$,  wins or $S = \emptyset$ and  wins

# A game where the angel wins

Consider $f = f_1 : \mathbb{R}^4 \to \mathbb{R}$:

$f_1(x_1, x_2, x_3, x_4) = 2 + 2x_1 + x_2 + 3x_3 + x_4$ and compute:

$$R_{\exists\forall\exists}(f) = \{z_1 \in \mathbb{R} \mid \exists x_1 \in [-1,1],\ \forall x_2 \in [-1,1],\ \forall x_4 \in [-1,1],\ \exists x_3 \in [-1,1],\ z_1 = f_1(x_1, x_2, x_3, x_4)\}$$

Let's play! - round $i = p = 4$


$$z_1 = [\, z_1^c - \delta_{x_3},\ z_1^c + \delta_{x_3} ] = [\, 2 - 3,\ 2 + 3]$$

where $z_1^c = f(0,0,0,0) = 2$

# A game where the angel wins

Consider $f = f_1 : \mathbb{R}^4 \to \mathbb{R}$:

$f_1(x_1, x_2, x_3, x_4) = 2 + 2x_1 + x_2 + 3x_3 + x_4$ and compute:

$$R_{\exists\forall\exists}(f) = \{z_1 \in \mathbb{R} \mid \exists x_1 \in [-1,1],\ \forall x_2 \in [-1,1],\ \forall x_4 \in [-1,1],\ \exists x_3 \in [-1,1],\ z_1 = f_1(x_1,x_2,x_3,x_4)\}$$

Let's play! - round $i = 3$

$$z_1 = [\ z_1^c \ +\delta_{x_4} \ -\delta_{x_3},\ z_1^c \ -\delta_{x_4} \ +\delta_{x_3}]$$
$$= [\ 2 \ +1 \ -3,\ 2 \ -1 \ +3]$$

Angel wins $1 - 3 < -1 + 3$!

# A game where the angel wins

Consider $f = f_1 : \mathbb{R}^4 \rightarrow \mathbb{R}$:

$f_1(x_1, x_2, x_3, x_4) = 2 + 2x_1 + x_2 + 3x_3 + x_4$ and compute:

$$R_{\exists\forall\exists}(f) = \{z_1 \in \mathbb{R} \mid \exists x_1 \in [-1,1],\ \forall x_2 \in [-1,1],\ \forall x_4 \in [-1,1],\ \exists x_3 \in [-1,1],\ z_1 = f_1(x_1,x_2,x_3,x_4)\}$$

Let's play! - round $i = 2$

$$\begin{aligned} z_1 &= [\ z_1^c\ +\delta_{x_2}\ +\delta_{x_4}\ -\delta_{x_3},\ z_1^c\ -\delta_{x_2}\ -\delta_{x_4}\ +\delta_{x_3}] \\ &= [\ 2\ +1\ +1\ -3,\ 2\ -1\ -1\ +3] \end{aligned}$$

Angel still wins $1 + 1 - 3 < -1 - 1 + 3$!

# A game where the angel wins

Consider $f = f_1 : \mathbb{R}^4 \to \mathbb{R}$:

$f_1(x_1, x_2, x_3, x_4) = 2 + 2x_1 + x_2 + 3x_3 + x_4$ and compute:

$$R_{\exists\forall\exists}(f) = \{z_1 \in \mathbb{R} \mid \exists x_1 \in [-1,1],\ \forall x_2 \in [-1,1],\ \forall x_4 \in [-1,1],\ \exists x_3 \in [-1,1],\ z_1 = f_1(x_1,x_2,x_3,x_4)\}$$

Let's play! - round $i = 1$

$$z_1 = [\, z_1^c \; -\delta_{x_1} \; +\delta_{x_2} \; +\delta_{x_4} \; -\delta_{x_3},\; z_1^c \; +\delta_{x_1} \; -\delta_{x_2} \; -\delta_{x_4} \; +\delta_{x_3}\,]$$
$$= [\; 2 \; -2 \; +1 \; +1 \; -3,\; 2 \; +2 \; -1 \; -1 \; +3] = [-1, 5]$$

Final win from the angel side: $-2 + 1 + 1 - 3 < 2 - 1 - 1 + 3$!

# Slightly changing the game so that the devil wins

Consider $f = f_1 : \mathbb{R}^4 \to \mathbb{R}$ again - exchanging the roles of $x_3$ and $x_4$:

$f_1(x_1, x_2, x_3, x_4) = 2 + 2x_1 + x_2 + 3x_3 + x_4$ but now compute:

$$R_{\exists\forall\exists}(f) = \{z_1 \in \mathbb{R} | \exists x_1 \in [-1, 1],\ \forall x_2 \in [-1, 1],\ \forall x_3 \in [-1, 1],\ \exists x_4 \in [-1, 1],\ z_1 = f_1(x_1, x_2, x_3, x_4)\}$$

Let's play! - round $i = m = 4$

$$\begin{aligned} z_1 &= [\ z_1^c \ -\delta_{x_4},\ z_1^c \ +\delta_{x_4}] \\ &= [\ 2 \ -1,\ 2 \ +1] \end{aligned}$$

where $z_1^c = f(0, 0, 0, 0) = 2$

# Slightly changing the game so that the devil wins

Consider $f = f_1 : \mathbb{R}^4 \to \mathbb{R}$ again - exchanging the roles of $x_3$ and $x_4$:

$f_1(x_1, x_2, x_3, x_4) = 2 + 2x_1 + x_2 + 3x_3 + x_4$ but now compute:

$$R_{\exists\forall\exists}(f) = \{z_1 \in \mathbb{R} | \exists x_1 \in [-1, 1],\ \forall x_2 \in [-1, 1],\ \forall x_3 \in [-1, 1],\ \exists x_4 \in [-1, 1],\ z_1 = f_1(x_1, x_2, x_3, x_4)\}$$

Let's play! - round $i = 3$

$$\begin{aligned} z_1 &= [\, z_1^c \ +\delta_{x_3} \ -\delta_{x_4},\ z_1^c \ -\delta_{x_3} \ +\delta_{x_4}] \\ &= [\ 2 \ +3 \ -1,\ 2 \ -3 \ +1] \end{aligned}$$

Demon wins $3 - 1 > -3 + 1$ and $S = \emptyset$!

# Slightly changing the game so that the devil wins

Consider $f = f_1 : \mathbb{R}^4 \to \mathbb{R}$ again - exchanging the roles of $x_3$ and $x_4$:

$f_1(x_1, x_2, x_3, x_4) = 2 + 2x_1 + x_2 + 3x_3 + x_4$ but now compute:

$$R_{\exists\forall\exists}(f) = \{z_1 \in \mathbb{R} | \exists x_1 \in [-1,1],\ \forall x_2 \in [-1,1],\ \forall x_3 \in [-1,1],\ \exists x_4 \in [-1,1],\ z_1 = f_1(x_1, x_2, x_3, x_4)\}$$

Let's play! - round $i = 3$

$$\begin{aligned} z_1 &= [\, z_1^c \ +\delta_{x_3} \ -\delta_{x_4},\ z_1^c \ -\delta_{x_3} \ +\delta_{x_4}] \\ &= [\ 2 \ +3 \ -1,\ 2 \ -3 \ +1] \end{aligned}$$

Demon wins $3 - 1 > -3 + 1$ and $S = \emptyset$!

The general formula, and its proof, in the paper E. Goubault, S. Putot: Inner and outer approximate quantifier elimination for general reachability problems. HSCC 2024

# Step 2, quantified reachability for scalar non-linear functions

# How do we find simple inner and outer-approximations of functions?

## Generalized mean-value theorem for $f: \mathbb{R}^m \to \mathbb{R}$

Suppose we can bound partial derivatives of $f$ by $\nabla_j = [\underline{\nabla}_j, \overline{\nabla}_j]$ $(i = 1, \ldots, p)$:

$$\left\{ \left| \frac{\partial f}{\partial x_j}(x_1, \ldots, x_i, 0, \ldots, 0) \right| \mid x_l \in [-1, 1],\ l = 1, \ldots, i \right\} \subseteq \nabla_j$$

## Then:

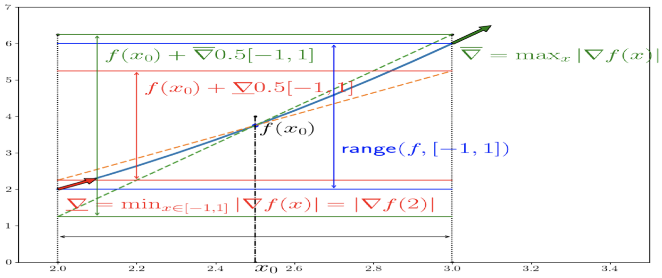
Writing inner and outer contributions: $I_i = \underline{\nabla}_j[-1, 1]$, $O_j = \overline{\nabla}_j[-1, 1]$, $j = 1, \ldots, p$ we get inner and outer-approximations of $f$:

$$f(0, \ldots, 0) + \sum_{i=1}^{p} I_i \subseteq f([-1, 1]^p) \subseteq f(0, \ldots, 0) + \sum_{i=1}^{p} O_i$$

See e.g. A. Goldsztejn. 2012. Modal Intervals Revisited, Part 2: A Generalized Interval Mean Value Extension. Reliable Computing 2012) and E. Goubault, S. Putot: Inner and outer reachability for the verification of control systems. HSCC 2019

# How do we find simple inner and outer-approximations of functions?

## Generalized mean-value theorem



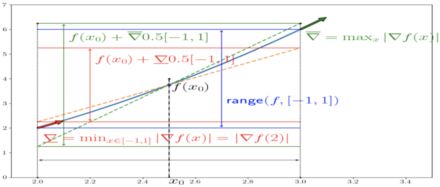

## Then:

Writing inner and outer contributions: $I_i = \underline{\nabla}_j[-1,1]$, $O_j = \overline{\nabla}_j[-1,1]$, $j = 1, \ldots, p$ we get inner and outer-approximations of $f$:

$$f(0,\ldots,0) + \sum_{i=1}^{p} I_i \subseteq f([-1,1]^p) \subseteq f(0,\ldots,0) + \sum_{i=1}^{p} O_i$$

# How do we find simple inner and outer-approximations of functions?

## Generalized mean-value theorem




## Then:

Writing inner and outer contributions: $I_i = \underline{\nabla}_j[-1,1]$, $O_j = \overline{\nabla}_j[-1,1]$, $j = 1, \ldots, p$ we get inner and outer-approximations of $f$:

$$f(0,\ldots,0) + \sum_{i=1}^{p} I_i \subseteq f([-1,1]^p) \subseteq f(0,\ldots,0) + \sum_{i=1}^{p} O_i$$

(other approximation methods, higher-order in particular, see e.g. Eric Goubault Sylvie Putot, "Tractable higher-order under-approximating AE extensions for non-linear systems" ADHS 2021)

# Example of inner-outer approximation by generalized mean value theorem

## Example, function $g : \mathbb{R}^3 \to \mathbb{R}$ on $[-1,1]^3$

$$g(x_1,x_2,x_3) = \frac{x_1^2}{4} + (x_2+1)(x_3+2) + (x_3+3)^2.$$

Compute inner and outer approximation of the range of $g$, i.e. of
$R_\exists(g) = \{z \mid \exists x_1 \in [-1,1],\ \exists x_2 \in [-1,1],\ \exists x_3 \in [-1,1], z = g(x_1,x_2,x_3)\}$

## Individual contributions of each argument

- $\nabla_1 = |\frac{\partial g}{\partial x_1}| = |\frac{x_1}{2}| \in \left[0, \frac{1}{2}\right]$, $\nabla_2 = |\frac{\partial g}{\partial x_2}| = |x_3+2| \in [1,3]$, $\nabla_3 = |\frac{\partial g}{\partial x_3}| = |x_2 + 1 + 2(x_3+3)| \in [4,10]$, and $c = g(0,0,0) = 11$.
- $O_1 = \left[-\frac{1}{2}, \frac{1}{2}\right]$, $I_1 = 0$, $O_2 = [-3,3]$, $I_2 = [-1,1]$ and $O_3 = [-10,10]$, $I_3 = [-4,4]$.

## Inner and outer-approximations

$$g(0,\ldots,0) + \sum_{i=1}^{p} I_i \subseteq g([-1,1]^p) \subseteq g(0,\ldots,0) + \sum_{i=1}^{p} O_i$$

# Example of inner-outer approximation by generalized mean value theorem

## Example, function $g : \mathbb{R}^3 \to \mathbb{R}$ on $[-1,1]^3$

$$g(x_1,x_2,x_3) = \frac{x_1^2}{4} + (x_2+1)(x_3+2) + (x_3+3)^2.$$

Compute inner and outer approximation of the range of $g$, i.e. of $R_\exists(g) = \{z \mid \exists x_1 \in [-1,1],\ \exists x_2 \in [-1,1],\ \exists x_3 \in [-1,1], z = g(x_1,x_2,x_3)\}$

## Individual contributions of each argument

- $\nabla_1 = |\frac{\partial g}{\partial x_1}| = |\frac{x_1}{2}| \in \left[0, \frac{1}{2}\right]$, $\nabla_2 = |\frac{\partial g}{\partial x_2}| = |x_3+2| \in [1,3]$, $\nabla_3 = |\frac{\partial g}{\partial x_3}| = |x_2+1+2(x_3+3)| \in [4,10]$, and $c = g(0,0,0) = 11$.
- $O_1 = \left[-\frac{1}{2}, \frac{1}{2}\right]$, $I_1 = 0$, $O_2 = [-3,3]$, $I_2 = [-1,1]$ and $O_3 = [-10,10]$, $I_3 = [-4,4]$.

## Inner and outer-approximations

$$11 + [-0,0] + [-1,1] + [-4,4] \subseteq g([-1,1]^p) \subseteq 11 + [-\frac{1}{2},\frac{1}{2}] + [-3,3] + [-10,10]$$

# Example of inner-outer approximation by generalized mean value theorem

## Example, function $g : \mathbb{R}^3 \to \mathbb{R}$ on $[-1,1]^3$

$$g(x_1, x_2, x_3) = \frac{x_1^2}{4} + (x_2+1)(x_3+2) + (x_3+3)^2.$$

Compute inner and outer approximation of the range of $g$, i.e. of
$R_\exists(g) = \{z \mid \exists x_1 \in [-1,1],\ \exists x_2 \in [-1,1],\ \exists x_3 \in [-1,1], z = g(x_1,x_2,x_3)\}$

## Individual contributions of each argument

- $\nabla_1 = |\frac{\partial g}{\partial x_1}| = |\frac{x_1}{2}| \in \left[0, \frac{1}{2}\right]$, $\nabla_2 = |\frac{\partial g}{\partial x_2}| = |x_3+2| \in [1,3]$,
  $\nabla_3 = |\frac{\partial g}{\partial x_3}| = |x_2 + 1 + 2(x_3+3)| \in [4,10]$, and $c = g(0,0,0) = 11$.
- $O_1 = \left[-\frac{1}{2}, \frac{1}{2}\right]$, $I_1 = 0$, $O_2 = [-3,3]$, $I_2 = [-1,1]$ and $O_3 = [-10,10]$, $I_3 = [-4,4]$.

## Inner and outer-approximations

$$[6,16] \subseteq g([-1,1]^p) \subseteq [-2.5, 24.5]$$

(real range $[4.25, 22.25]$)

# Let us play a slightly more involved two-player game!

$f$ is the non-linear function with $I_i$ and $O_j$ for each variable $x_i$, either quantified by $Q_i$ being $\exists$ or by $\forall$.

The players, again!

($\exists$-player)

($\forall$-player)

# Let us play a slightly more involved two-player game!

$f$ is the non-linear function with $I_i$ and $O_j$ for each variable $x_i$, either quantified by $Q_i$ being $\exists$ or by $\forall$.

## The rules of the outer-approximation game

Compute $S = [\underline{S}, \overline{S}]$, interval outer-approximating the quantified reachable set; initially $S = \{f(0, \ldots, 0)\}$

- At round $i$ from $p$ to 1,  plays if $Q_i = \exists$,  plays if $Q_i = \forall$
  -  widens $S$ by the maximal contribution $O_i$ ($\underline{S}$-=$\underline{O}_i$, $\overline{S}$+=$\overline{O}_i$)
  -  shrink $S$ by the minimal contribution $I_i$ ($\underline{S}$+=$\underline{I}_i$, $\overline{S}$-=$\overline{I}_i$)
- Stops either after step $i = 1$,  wins or $S = \emptyset$ and  wins

# Let us play a slightly more involved two-player game!

$f$ is the non-linear function with $I_i$ and $O_j$ for each variable $x_i$, either quantified by $Q_i$ being $\exists$ or by $\forall$.

## The rules of the inner-approximation game

Compute $S = [\underline{S}, \overline{S}]$, interval inner-approximating the quantified reachable set; initially $S = \{f(0, \ldots, 0)\}$

- At round $i$ from $p$ to 1,  plays if $Q_i = \exists$,  plays if $Q_i = \forall$
  -  widens $S$ by the minimal contribution $I_i$ ($\underline{S}$-=$\underline{I}_i$, $\overline{S}$+=$\overline{I}_i$)
  -  shrink $S$ by the maximal contribution $O_i$ ($\underline{S}$+=$\underline{O}_i$, $\overline{S}$-=$\overline{O}_i$)
- Stops either after step $i = 1$,  wins or $S = \emptyset$ and  wins

# Let us play a slightly more involved two-player game!

$f$ is the non-linear function with $I_i$ and $O_j$ for each variable $x_i$, either quantified by $Q_i$ being $\exists$ or by $\forall$.

## The rules of the inner-approximation game

Compute $S = [\underline{S}, \overline{S}]$, interval inner-approximating the quantified reachable set; initially $S = \{f(0, \ldots, 0)\}$

- At round $i$ from $p$ to 1,  plays if $Q_i = \exists$,  plays if $Q_i = \forall$
  -  widens $S$ by the minimal contribution $I_i$ ($\underline{S}$-=$\underline{I}_i$, $\overline{S}$+=$\overline{I}_i$)
  -  shrink $S$ by the maximal contribution $O_i$ ($\underline{S}$+=$\underline{O}_i$, $\overline{S}$-=$\overline{O}_i$)
- Stops either after step $i = 1$,  wins or $S = \emptyset$ and  wins

Formalized theorem and proof in the paper E. Goubault, S. Putot: Inner and outer approximate quantifier elimination for general reachability problems. HSCC 2024

# An outer-approximation game

## Example, function $g : \mathbb{R}^3 \to \mathbb{R}$ on $[-1,1]^3$

$$g(x_1, x_2, x_3) = \frac{x_1^2}{4} + (x_2+1)(x_3+2) + (x_3+3)^2.$$

Compute $R_{\exists\forall\exists}(g) = \{z \mid \exists x_1 \in [-1,1],\ \forall x_2 \in [-1,1],\ \exists x_3 \in [-1,1], z = g(x_1,x_2,x_3)\}$

## "Individual contributions" of each argument

- $\nabla_1 = |\frac{\partial g}{\partial x_1}| = |\frac{x_1}{2}| \in \left[0, \frac{1}{2}\right]$, $\nabla_2 = |\frac{\partial g}{\partial x_2}| = |x_3+2| \in [1,3]$, $\nabla_3 = |\frac{\partial g}{\partial x_3}| = |x_2 + 1 + 2(x_3+3)| \in [4,10]$, and $c = g(0,0,0) = 11$.
- $O_1 = \left[-\frac{1}{2}, \frac{1}{2}\right]$, $I_1 = 0$, $O_2 = [-3,3]$, $I_2 = [-1,1]$ and $O_3 = [-10,10]$, $I_3 = [-4,4]$.

## Outer-approximation of $R_{\exists\forall\exists}(g)$ - round 3


$$\begin{array}{llllll} [ & c & +\underline{O}_3, & c & +\overline{O}_3 & ] \\ = [ & 11 & -10, & 11 & +10 & ] \end{array}$$

# An outer-approximation game

## Example, function $g : \mathbb{R}^3 \to \mathbb{R}$ on $[-1,1]^3$

$$g(x_1, x_2, x_3) = \frac{x_1^2}{4} + (x_2+1)(x_3+2) + (x_3+3)^2.$$

Compute $R_{\exists\forall\exists}(g) = \{z \mid \exists x_1 \in [-1,1],\ \forall x_2 \in [-1,1],\ \exists x_3 \in [-1,1], z = g(x_1,x_2,x_3)\}$

## "Individual contributions" of each argument

- $\nabla_1 = |\frac{\partial g}{\partial x_1}| = |\frac{x_1}{2}| \in [0, \frac{1}{2}]$, $\nabla_2 = |\frac{\partial g}{\partial x_2}| = |x_3 + 2| \in [1,3]$,
  $\nabla_3 = |\frac{\partial g}{\partial x_3}| = |x_2 + 1 + 2(x_3+3)| \in [4, 10]$, and $c = g(0,0,0) = 11$.
- $O_1 = [-\frac{1}{2}, \frac{1}{2}]$, $I_1 = 0$, $O_2 = [-3,3]$, $I_2 = [-1,1]$ and $O_3 = [-10,10]$, $I_3 = [-4,4]$.

## Outer-approximation of $R_{\exists\forall\exists}(g)$ - round 2

$$\begin{array}{l} [\ c \ +\overline{I}_2 \ +\underline{O}_3,\ c \ +\underline{I}_2 \ +\overline{O}_3\ ] \\ = [\ 11 \ +1 \ -10,\ 11 \ -1 \ +10\ ] \end{array}$$

# An outer-approximation game

**Example, function $g : \mathbb{R}^3 \to \mathbb{R}$ on $[-1,1]^3$**

$$g(x_1, x_2, x_3) = \frac{x_1^2}{4} + (x_2+1)(x_3+2) + (x_3+3)^2.$$

Compute $R_{\exists\forall\exists}(g) = \{z \mid \exists x_1 \in [-1,1],\ \forall x_2 \in [-1,1],\ \exists x_3 \in [-1,1], z = g(x_1,x_2,x_3)\}$

**"Individual contributions" of each argument**

- $\nabla_1 = |\frac{\partial g}{\partial x_1}| = |\frac{x_1}{2}| \in \left[0, \frac{1}{2}\right]$, $\nabla_2 = |\frac{\partial g}{\partial x_2}| = |x_3 + 2| \in [1,3]$, $\nabla_3 = |\frac{\partial g}{\partial x_3}| = |x_2 + 1 + 2(x_3+3)| \in [4,10]$, and $c = g(0,0,0) = 11$.
- $O_1 = \left[-\frac{1}{2}, \frac{1}{2}\right]$, $I_1 = 0$, $O_2 = [-3,3]$, $I_2 = [-1,1]$ and $O_3 = [-10,10]$, $I_3 = [-4,4]$.

**Outer-approximation of $R_{\exists\forall\exists}(g)$ - round 1, Angel wins**

$$\begin{array}{ccccccccccc} & [ & c & +\underline{O}_1 & +\overline{I}_2 & +\underline{O}_3, & c & +\overline{O}_1 & +\underline{I}_2 & +\overline{O}_3 & ] \\ = & [ & 11 & -\frac{1}{2} & +1 & -10, & 11 & +\frac{1}{2} & -1 & +10 & ] & = [1.5, 20.5] \end{array}$$

# An outer-approximation game

## Example, function $g : \mathbb{R}^3 \to \mathbb{R}$ on $[-1,1]^3$

$$g(x_1, x_2, x_3) = \frac{x_1^2}{4} + (x_2+1)(x_3+2) + (x_3+3)^2.$$

Compute $R_{\exists\forall\exists}(g) = \{z \mid \exists x_1 \in [-1,1],\ \forall x_2 \in [-1,1],\ \exists x_3 \in [-1,1], z = g(x_1,x_2,x_3)\}$

## "Individual contributions" of each argument

- $\nabla_1 = |\frac{\partial g}{\partial x_1}| = |\frac{x_1}{2}| \in \left[0, \frac{1}{2}\right]$, $\nabla_2 = |\frac{\partial g}{\partial x_2}| = |x_3+2| \in [1,3]$, $\nabla_3 = |\frac{\partial g}{\partial x_3}| = |x_2 + 1 + 2(x_3+3)| \in [4,10]$, and $c = g(0,0,0) = 11$.
- $O_1 = \left[-\frac{1}{2}, \frac{1}{2}\right]$, $I_1 = 0$, $O_2 = [-3,3]$, $I_2 = [-1,1]$ and $O_3 = [-10,10]$, $I_3 = [-4,4]$.

## Outer-approximation of $R_{\exists\forall\exists}(g)$ - round 1, Angel wins

$$\begin{array}{l} [\ c + \underline{O}_1 + \overline{I}_2 + \underline{O}_3,\ c + \overline{O}_1 + \underline{I}_2 + \overline{O}_3\ ] \\ = [\ 11 - \frac{1}{2} + 1 - 10,\ 11 + \frac{1}{2} - 1 + 10\ ] = [1.5, 20.5] \end{array}$$

(in comparison, the sampling based estimation is $[6.25, 16.25]$)

# An inner-approximation game

**Example, function $g : \mathbb{R}^3 \to \mathbb{R}$ on $[-1,1]^3$**

Compute $R_{\exists\forall\exists}(g) = \{z \mid \exists x_1 \in [-1,1],\ \forall x_2 \in [-1,1],\ \exists x_3 \in [-1,1], z = g(x_1, x_2, x_3)\}$.

**"Individual contributions" of each argument**

- $\nabla_1 = |\frac{\partial g}{\partial x_1}| = |\frac{x_1}{2}| \in \left[0, \frac{1}{2}\right]$, $\nabla_2 = |\frac{\partial g}{\partial x_2}| = |x_3 + 2| \in [1, 3]$, $\nabla_3 = |\frac{\partial g}{\partial x_3}| = |x_2 + 1 + 2(x_3 + 3)| \in [4, 10]$, and $c = g(0,0,0) = 11$.
- $O_1 = \left[-\frac{1}{2}, \frac{1}{2}\right]$, $I_1 = 0$, $O_2 = [-3, 3]$, $I_2 = [-1, 1]$ and $O_3 = [-10, 10]$, $I_3 = [-4, 4]$.

**Inner-approximation of $R_{\exists\forall\exists}(g)$ - round 3**


$$[\ c + \underline{I}_3,\ c + \overline{I}_3\ ]$$
$$= [\ 11 - 4,\ 11 + 4\ ]$$

# An inner-approximation game

**Example, function $g : \mathbb{R}^3 \to \mathbb{R}$ on $[-1,1]^3$**

Compute $R_{\exists\forall\exists}(g) = \{z \mid \exists x_1 \in [-1,1],\ \forall x_2 \in [-1,1],\ \exists x_3 \in [-1,1], z = g(x_1,x_2,x_3)\}$.

**"Individual contributions" of each argument**

- $\nabla_1 = |\frac{\partial g}{\partial x_1}| = |\frac{x_1}{2}| \in \left[0, \frac{1}{2}\right]$, $\nabla_2 = |\frac{\partial g}{\partial x_2}| = |x_3 + 2| \in [1,3]$, $\nabla_3 = |\frac{\partial g}{\partial x_3}| = |x_2 + 1 + 2(x_3+3)| \in [4,10]$, and $c = g(0,0,0) = 11$.
- $O_1 = \left[-\frac{1}{2}, \frac{1}{2}\right]$, $I_1 = 0$, $O_2 = [-3,3]$, $I_2 = [-1,1]$ and $O_3 = [-10,10]$, $I_3 = [-4,4]$.

**Inner-approximation of $R_{\exists\forall\exists}(g)$ - round 2**

$$\begin{array}{l} [\quad c \quad +\overline{O}_2 \quad +\underline{I}_3, \quad c \quad +\underline{O}_2 \quad +\overline{I}_3 \quad ] \\ = [\quad 11 \quad +3 \quad -4, \quad 11 \quad -3 \quad +4 \quad ] \end{array}$$

# An inner-approximation game

**Example, function $g : \mathbb{R}^3 \to \mathbb{R}$ on $[-1,1]^3$**

Compute $R_{\exists\forall\exists}(g) = \{z \mid \exists x_1 \in [-1,1],\ \forall x_2 \in [-1,1],\ \exists x_3 \in [-1,1], z = g(x_1,x_2,x_3)\}$.

**"Individual contributions" of each argument**

- $\nabla_1 = |\frac{\partial g}{\partial x_1}| = |\frac{x_1}{2}| \in \left[0, \frac{1}{2}\right]$, $\nabla_2 = |\frac{\partial g}{\partial x_2}| = |x_3 + 2| \in [1,3]$, $\nabla_3 = |\frac{\partial g}{\partial x_3}| = |x_2 + 1 + 2(x_3+3)| \in [4,10]$, and $c = g(0,0,0) = 11$.
- $O_1 = \left[-\frac{1}{2}, \frac{1}{2}\right]$, $I_1 = 0$, $O_2 = [-3,3]$, $I_2 = [-1,1]$ and $O_3 = [-10,10]$, $I_3 = [-4,4]$.

**Inner-approximation of $R_{\exists\forall\exists}(g)$ - round 1, Angel wins**

$$\begin{array}{l} [\quad c \ + \underline{I}_1 \ + \overline{O}_2 \ + \underline{I}_3, \quad c \ + \overline{I}_1 \ + \underline{O}_2 \ + \overline{I}_3 \quad ] \\ = [\ 11 \quad 0 \quad +3 \quad -4, \quad 11 \quad +0 \quad -3 \quad +4 \ ] = [10, 12] \end{array}$$

# An inner-approximation game

**Example, function $g : \mathbb{R}^3 \to \mathbb{R}$ on $[-1,1]^3$**

Compute $R_{\exists\forall\exists}(g) = \{z \mid \exists x_1 \in [-1,1],\ \forall x_2 \in [-1,1],\ \exists x_3 \in [-1,1], z = g(x_1,x_2,x_3)\}$.

**"Individual contributions" of each argument**

- $\nabla_1 = |\frac{\partial g}{\partial x_1}| = |\frac{x_1}{2}| \in \left[0, \frac{1}{2}\right]$, $\nabla_2 = |\frac{\partial g}{\partial x_2}| = |x_3 + 2| \in [1,3]$, $\nabla_3 = |\frac{\partial g}{\partial x_3}| = |x_2 + 1 + 2(x_3+3)| \in [4,10]$, and $c = g(0,0,0) = 11$.
- $O_1 = \left[-\frac{1}{2}, \frac{1}{2}\right]$, $I_1 = 0$, $O_2 = [-3,3]$, $I_2 = [-1,1]$ and $O_3 = [-10,10]$, $I_3 = [-4,4]$.

**Inner-approximation of $R_{\exists\forall\exists}(g)$ - round 1, Angel wins**

$$[\ c + \underline{I}_1 + \overline{O}_2 + \underline{I}_3,\ c + \overline{I}_1 + \underline{O}_2 + \overline{I}_3\ ]$$
$$= [\ 11 \quad 0 \quad +3 \quad -4,\ 11 \quad +0 \quad -3 \quad +4\ ] = [10,12]$$

(in comparison, the sampling based estimation is $[6.25, 16.25]$)

# Step 3, quantified reachability for general functions

# The problem with joint inner-approximations

## A simple example

Consider $f = (f_1, f_2) : \mathbb{R}^4 \to \mathbb{R}^2$:

$$\begin{array}{rcl} f_1(x_1, x_2, x_3, x_4) & = & 2 + 2x_1 + x_2 + 3x_3 + x_4 \\ f_2(x_1, x_2, x_3, x_4) & = & -1 - x_1 - x_2 + x_3 + 5x_4 \end{array}$$

$$R_{\exists\forall\exists}(f) = \{z \in \mathbb{R}^2 | \exists x_1 \in [-1,1],\ \forall x_2 \in [-1,1],\ \exists x_3 \in [-1,1],\ \exists x_4 \in [-1,1],\ z = f(x_1, x_2, x_3, x_4)\}$$

## Problem?

- Outer-approximation of each component $\Rightarrow$ outer-approximation of $R_{\exists\forall\exists}(f)$: Same calculation as before, 1 component at a time: $R_{\exists\forall\exists}(f) \subseteq [-3,7] \times [-7,5]$.
- Would find here as well $[-3,7] \times [-7,5] \subseteq R_{\exists\forall\exists}(f)$, wrong!

Reason: a witness for $\exists x_i$ may not be the same for each component of $f$!

# A solution for joint inner-approximations

## A simple relaxation

- Conjunction of quantified formulas for each component if no variable is existentially quantified in several components.
- Transform the quantified formula by strengthening them for that objective

## For example ($\underline{\forall}$ as relaxations of $\exists$):

$$\boxed{\exists x_1},\ \forall x_2,\ \underline{\forall} x_4,\ \boxed{\exists x_3},\ z_1 = f_1(x_1, x_2, x_3, x_4)$$
$$\underline{\forall} x_1,\ \forall x_2,\ \underline{\forall} x_3,\ \boxed{\exists x_4},\ z_2 = f_2(x_1, x_2, x_3, x_4)$$

(for each $i$, $\boxed{\exists x_i}$ appears in only one component of $f$)

# Example

Consider $f = (f_1, f_2) : \mathbb{R}^4 \to \mathbb{R}^2$:

$$\begin{array}{rcl} f_1(x_1, x_2, x_3, x_4) & = & 2 + 2x_1 + x_2 + 3x_3 + x_4 \\ f_2(x_1, x_2, x_3, x_4) & = & -1 - x_1 - x_2 + x_3 + 5x_4 \end{array}$$

$$R_{\exists\forall\exists}(f) = \{z \in \mathbb{R}^2 | \exists x_1 \in [-1,1],\ \forall x_2 \in [-1,1],\ \exists x_3 \in [-1,1],\ \exists x_4 \in [-1,1],\ z = f(x_1,x_2,x_3,x_4)\}$$

Same calculation as before, 1 component at a time: $R_{\exists\forall\exists}(f) \subseteq [-3,7] \times [-7,5]$.

For the joint inner-approximation, interpret (we already did the first component!):

$$\boxed{\exists x_1},\ \forall x_2,\ \forall x_4,\ \boxed{\exists x_3},\ z_1 = f_1(x_1,x_2,x_3,x_4)$$
$$\forall x_1,\ \forall x_2,\ \forall x_3,\ \boxed{\exists x_4},\ z_2 = f_2(x_1,x_2,x_3,x_4)$$

$$\begin{array}{rl} z_1 = & [\ z_1^c - \delta_{x_1} + \delta_{x_2} + \delta_{x_4} - \delta_{x_3},\ z_1^c + \delta_{x_1} - \delta_{x_2} - \delta_{x_4} + \delta_{x_3}] \\ = & [\ 2 - 2 + 1 + 1 - 3,\ 2 + 2 - 1 - 1 + 3] = [-1, 5] \end{array}$$

# Example

Consider $f = (f_1, f_2) : \mathbb{R}^4 \rightarrow \mathbb{R}^2$:

$$\begin{array}{rcl} f_1(x_1, x_2, x_3, x_4) & = & 2 + 2x_1 + x_2 + 3x_3 + x_4 \\ f_2(x_1, x_2, x_3, x_4) & = & -1 - x_1 - x_2 + x_3 + 5x_4 \end{array}$$

$$R_{\exists\forall\exists}(f) = \{z \in \mathbb{R}^2 | \exists x_1 \in [-1,1],\ \forall x_2 \in [-1,1],\ \exists x_3 \in [-1,1],\ \exists x_4 \in [-1,1],\ z = f(x_1, x_2, x_3, x_4)\}$$

For the joint inner-approximation, interpret (2nd component):

$$\boxed{\exists x_1},\ \forall x_2,\ \forall x_4,\ \boxed{\exists x_3},\ z_1 = f_1(x_1, x_2, x_3, x_4)$$

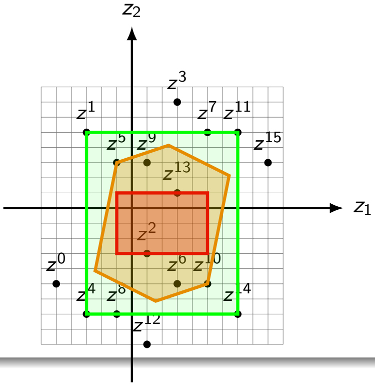
$$\forall x_1,\ \forall x_2,\ \forall x_3,\ \boxed{\exists x_4},\ z_2 = f_2(x_1, x_2, x_3, x_4)$$

$$\begin{array}{l} z_2 = [\ z_2^c\ + \delta_{x_1} + \delta_{x_2} + \delta_{x_4} - \delta_{x_3},\ z_1^c\ - \delta_{x_1} - \delta_{x_2} - \delta_{x_4} + \delta_{x_3}] \\ \quad = [\ -1\ \ +1+1+1\ \ -5,\ -1\ \ -1-1-1\ \ +5] = [-3, 1] \end{array}$$

Hence $[-1, 5] \times [-3, 1] \subseteq R_{\exists\forall\exists}(f) \subseteq [-3, 7] \times [-7, 5]$.

# Example, in picture

$R_{\exists\forall\exists}(f) = \{z \in \mathbb{R}^2 | \exists x_1,\ \forall x_2,\ \exists x_3, \exists x_4,\ z = f(x_1, x_2, x_3, x_4)\}$




- Samples $z^0, \ldots, z^{15}$ of $f([-1, 1]^4)$
- Outer-approximation (our method)
- Exact set $R_{\exists\forall\exists}(f)$
- Inner-approximation (our method)

# Applications to control systems

# Application to control systems

## Continuous time dynamical systems

- Contrarily to QE, method applicable directly on solutions of an ODE
- The inner and outer contributions, per variable $I_i$ and $O_i$ can be derived directly by guaranteed integration (e.g. Taylor models) on the corresponding variational ODE

# Example

## Dubbins vehicle

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} v cos(\theta) + b_1 \\ v sin(\theta) \\ a \end{pmatrix}$$

- Control period of $t = 0.5$, linear velocity $v = 1$,
- Initial conditions:
  $\mathbb{X}_0 = \{(x, y, \theta) \mid x \in [-0.1, 0.1],\ y \in [-0.1, 0.1],\ \theta \in [-0.01, 0.01]\}$,
- Control $a$ (angular velocity) in $\mathbb{U} = [-0.01, 0.01]$,
- disturbance $b_1$ in $\mathbb{W} = [-0.01, 0.01]$

We want to estimate (robust reachability):

$$R_{\exists\forall\exists}(\varphi) = \{z \in \mathbb{R}^m \mid \exists u \in \mathbb{U},\ \exists x_0 \in \mathbb{X}_0,\ \forall w \in \mathbb{W},\ \exists s \in [0, T],\ z = \varphi(s; x_0, u, w)\}$$

# Dubbins example

Direct computation from the ODE (no need for Taylor approximant here)

- Outer-approximation of a "central trajectory" $(x_c, y_c, \theta_c)$ starting at $x = 0$, $y = 0$, $\theta = 0$, $b_1 = 0$ and $a = 0$: $x_c = t$, $y_c = 0$ and $\theta_c = 0$,
- $\frac{\partial x}{\partial t} = cos(\theta) + b_1 \in [0.989999965, 1.01]$ hence $I_{x,t} = [0, 0.494999982]$, $O_{x,t} = [0, 0.505]$,
- Similarly for the other variables: $I_{y,t} = 0$, $O_{y,t} = [-sin(0.015)/2, sin(0.015)/2] = [-1.309\ 10^{-4}, 1.309\ 10^{-4}]$ and $I_{\theta,t} = 0$, $O_{\theta,t} = [-0.005, 0.005]$,
- The Jacobian of $\varphi$ with respect to $x_0$, $y_0$, $\theta_0$, $b_1$ and $a$, satisfies a variational equation: E.g.:

$$\dot{\left(\frac{\partial x}{\partial x_0}\right)} = -vsin(\theta)\frac{\partial \theta}{\partial x_0} + \frac{\partial b_1}{\partial x_0}$$

with $\frac{\partial x}{\partial x_0}(t = 0) = 1$, $\frac{\partial \theta}{\partial x_0}(t = 0) = 0$ etc.

# Dubbins example

## Direct computation from the ODE (no need for Taylor approximant here)

- Outer-approximation of a "central trajectory" $(x_c, y_c, \theta_c)$ starting at $x = 0$, $y = 0$, $\theta = 0$, $b_1 = 0$ and $a = 0$: $x_c = t$, $y_c = 0$ and $\theta_c = 0$,
- $\frac{\partial x}{\partial t} = cos(\theta) + b_1 \in [0.989999965, 1.01]$ hence $I_{x,t} = [0, 0.494999982]$, $O_{x,t} = [0, 0.505]$,
- Similarly for the other variables: $I_{y,t} = 0$, $O_{y,t} = [-sin(0.015)/2, sin(0.015)/2] = [-1.309\ 10^{-4}, 1.309\ 10^{-4}]$ and $I_{\theta,t} = 0$, $O_{\theta,t} = [-0.005, 0.005]$,
- The Jacobian of $\varphi$ with respect to $x_0$, $y_0$, $\theta_0$, $b_1$ and $a$, satisfies a variational equation:
  - $I_{x,a} = 0$, $O_{x,a} = [-6.545\ 10^{-7}, 6.545\ 10^{-7}]$, $I_{x,x_0} = O_{x,x_0} = [-0.1, 0.1]$, $I_{x,\theta_0} = 0$, $O_{x,\theta_0} = [-1.309\ 10^{-6}, 1.309\ 10^{-6}]$, $I_{x,b_1} = 0$, $O_{x,b_1} = [-0.005, 0.005]$,
  - $I_{y,a} = 0$, $O_{y,a} = [-0, 0025,\ 0.0025]$, $I_{y,y_0} = O_{y,y_0} = [-0.1, 0.1]$, $I_{y,\theta_0} = 0$, $O_{y,\theta_0} = [-0, 005, 0.005]$,
  - $I_{\theta,\theta_0} = O_{\theta,\theta_0} = [-0.01, 0.01]$, $I_{\theta,a} = 0$, $O_{\theta,a} = [0, 0.005]$,

# Dubbins example

## Compute $R_{\exists\forall\exists}$:

$$\exists a \in [-0.01, 0.01],\ \exists x_0 \in [-0.1, 0.1],\ \exists y_0 \in [-0.1, 0.1],\ \exists \theta_0 \in [-0.01, 0.01],\ \forall b_1 \in [-0.01, 0.01],\ \exists t \in [0, 0.5],\ z = \varphi(t; x_0, y_0, \theta_0, a, b_1)$$

## Hence, inner-approximation

Lower bound inner-approximation for $x$:

$$\begin{array}{ccccccc} x_c & +\underline{I}_{x,a} & +\underline{I}_{x,x_0} & +\underline{I}_{x,y_0} & +\underline{I}_{x,\theta_0} & +\overline{O}_{x,b_1} & +\underline{I}_{x,t} \\ = 0 & -0 & -0.1 & +0 & -0 & +0.005 & +0 \end{array}$$

which is equal to -0.095, and its upper bound:

$$\begin{array}{ccccccc} x_c & +\overline{I}_{x,a} & +\overline{I}_{x,x_0} & +\overline{I}_{x,y_0} & +\overline{I}_{x,\theta_0} & +\underline{O}_{x,b_1} & +\overline{I}_{x,t} \\ 0 & +0 & +0.1 & +0 & +0 & -0.005 & +0.494999982 \end{array}$$

which is equal to 0.589999982. Therefore the inner-approximation for $x$ is equal to $[-0.095, 0.589999982]$.

# Dubbins example

## Compute $R_{\exists\forall\exists}$:

$$\exists a \in [-0.01, 0.01],\ \exists x_0 \in [-0.1, 0.1],\ \exists y_0 \in [-0.1, 0.1],\ \exists \theta_0 \in [-0.01, 0.01],\ \forall b_1 \in [-0.01, 0.01],\ \exists t \in [0, 0.5],\ z = \varphi(t; x_0, y_0, \theta_0, a, b_1)$$

## Hence, outer-approximation

Lower bound outer-approximation for the $x$:

$$\begin{array}{ccccccc} x_c & +\underline{O}_{x,a} & +\underline{O}_{x,x_0} & +\underline{O}_{x,y_0} & +\underline{O}_{x,\theta_0} & +\overline{I}_{x,b_1} & +\underline{O}_{x,t} \\ = 0 & -6.545\ 10^{-7} & -0.1 & +0 & -1.309\ 10^{-6} & +0 & +0 \end{array}$$

which is equal to -0.1000019635, and its upper bound:

$$\begin{array}{ccccccc} x_c & +\overline{O}_{x,a} & +\overline{O}_{x,x_0} & +\overline{O}_{x,y_0} & +\overline{O}_{x,\theta_0} & +\underline{I}_{x,b_1} & +\overline{O}_{x,t} \\ = 0 & +6.545\ 10^{-7} & +0.1 & 0 & +1.309\ 10^{-6} & -0 & +0.505 \end{array}$$

which is equal to 0.6050019635. Therefore the outer-approximation for $x$ is equal to $[-0.1000019635, 0.6050019635]$.

# Dubbins example

Compute $R_{\exists\forall\exists}$:

$$\exists a \in [-0.01, 0.01],\ \exists x_0 \in [-0.1, 0.1],\ \exists y_0 \in [-0.1, 0.1],\\ \exists \theta_0 \in [-0.01, 0.01],\ \forall b_1 \in [-0.01, 0.01],\ \exists t \in [0, 0.5],\\ z = \varphi(t; x_0, y_0, \theta_0, a, b_1)$$

And...

- for $y$ the inner-approximation $[-0.1, 0.1]$ and over-approximation $[-0.1076309, 0.1076309]$,
- and for $\theta$ the inner-approximation $[-0.01,\ 0.01]$ and over-approximation $[-0.02, 0.02]$.

Very close to results obtained by quantifier elimination (Mathematica), here with a much smaller complexity.

# Last application: Dubbins!

## Space relaxation

$$R_{\exists\forall\exists}(\varphi) = \{(x, y, \theta) \mid \exists a \in [-0.01, 0.01],\ \exists x_0 \in [-0.1, 0.1],\\ \exists y_0 \in [-0.1, 0.1],\ \exists \theta_0 \in [-0.01, 0.01],\ \forall b_1 \in [-0.01, 0.01],\\ \exists t \in [0, 0.5],\ \exists \delta_2 \in [-1.309\ 10^{-4}, 1.309\ 10^{-4}],\ \exists \delta_3 \in [-0.005, 0.005],\\ (x, y, \theta) = \varphi(t; x_0, y_0, \theta_0, a, b_1) + (0, \delta_2, \delta_3)\}$$

## Outer-approximation

$$R_{\exists\forall\exists}(\varphi) \subseteq [-0.1000019635, 0.6050019635] \times\\ [0.1077618, 0.1077618] \times [-0.025, 0.025]$$

# Last application: Dubbins!

$$R_{\exists\forall\exists}(\varphi) = \{(x,y,\theta) \mid \exists a \in [-0.01, 0.01],\ \exists x_0 \in [-0.1, 0.1],\ \exists y_0 \in [-0.1, 0.1],\ \exists \theta_0 \in [-0.01, 0.01],\ \forall b_1 \in [-0.01, 0.01],\ \exists t \in [0, 0.5],\ \exists \delta_2 \in [-1.309\ 10^{-4}, 1.309\ 10^{-4}],\ \exists \delta_3 \in [-0.005, 0.005],\ (x,y,\theta) = \varphi(t; x_0, y_0, \theta_0, a, b_1) + (0, \delta_2, \delta_3)\}$$

For the inner-approximation, interpret:

$$\forall a,\ \forall y_0, \forall \theta_0,\ \boxed{\exists x_0},\ \forall b_1,\ \forall \delta_2,\ \forall \delta_3,\ \boxed{\exists t},\ x = \varphi_x(t; x_0, y_0, \theta_0, a, b_1)$$

$$\forall a,\ \forall x_0, \forall \theta_0,\ \boxed{\exists y_0},\ \forall b_1,\ \forall \delta_3,\ \forall t,\ \boxed{\exists \delta_2},\ y = \varphi_y(t; x_0, y_0, \theta_0, a, b_1) + \delta_2$$

$$\forall x_0,\ \forall y_0, \boxed{\exists \theta_0,\ \exists a},\ \forall b_1,\ \forall \delta_2,\ \forall t,\ \boxed{\exists \delta_3},\ \theta = \varphi_\theta(t; x_0, y_0, \theta_0, a, b_1) + \delta_3$$

$$[-0.0949993455, 0.5899993275] \times [-0.0925, 0.0925] \times [-0.01, 0.01] \subseteq R_{\exists\forall\exists}(\varphi)$$

(timeout using quantifier elimination under Mathematica)

# To conclude

### Implementation

In `Julia`, using the packages `LazySets` for manipulating boxes (`Hyperrectangles`) and `Symbolics` for automatic differentiation.

### Performances

- Benchmarks on a Macbook Pro 2.3GHz Intel core i9 with 8 cores, measuring timings using the `Benchmark Julia` package.
- On a variety of problems up to 2000 variables in the linear case, 104 variables in the non-linear case, shows excellent performance (and QE cannot solve some of the problems with more than 10 variables even in a very long time)

More in the paper E. Goubault, S. Putot: Inner and outer approximate quantifier elimination for general reachability problems. HSCC 2024

# Thanks!

### More developments soon

with approximations of full QE/quantified constrained solving, and higher-order set-based methods

### Any questions?

{eric.goubault,sylvie.putot}@polytechnique.edu