Positivity proofs for linear recurrences

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I. Positivity Problem

Examples

- [Straub, Zudilin 2015] For all $n \in \mathbb{N}$,

$$s_n = \sum_{k=0}^{n} (-27)^{n-k} 2^{2k-n} \frac{(3k)!}{k!^3} \binom{k}{n-k} \geq 0$$

$$2(n+2)^2 s_{n+2} = 3(27n^2 + 81n + 62) s_{n+1} - 81(3n+2)(3n+4) s_n, \quad s_0 = 1,\ s_1 = 12$$

Goal: Prove the positivity using the recurrence relation

- [Gillis, Reznick, Zeilberger 83] For all $r \geq 4$,

$$\frac{1}{1-\sum_{i=1}^{r} t_i + r!\prod_{i=1}^{r} t_i} = \sum \mathbf{d}_{\mathbf{i_1},\mathbf{i_2},\ldots,\mathbf{i_r}}\; t_1^{i_1} \cdots t_r^{i_r}$$

For $n \in \mathbb{N}$, $d_{n,n,\ldots,n} \geq 0$

Automatic proof up to order 6 [Kaures 2007]

Automatic proof up to $r = 17$ [Pillwein 2019]

Conjecture proved [Yu 2019]

P-finite sequences

- $(u_n)_{n\in\mathbb{N}}$ is **P-finite** of **order** d if it is a solution of

$$p_d(n)u_{n+d} = p_{d-1}(n)u_{n+d-1} + \cdots + p_0(n)u_n,\ n \in \mathbb{N}$$

$p_i \in \mathbb{Q}[X]$

$p_d \neq 0$

- If the $(p_i)_{i=0}^d$ are constant, $(u_n)_{n\in\mathbb{N}}$ is **C-finite.**

Closure property:

$(u_n)_n,\ (v_n)_n$ P-finite (C-finite) $\Longrightarrow$ $(u_nv_n)_n, (u_n + v_n)_n$ P-finite (C-finite)

Positivity Problem

Positivity problem:

Input: $p_d(n)u_{n+d} = p_{d-1}(n)u_{n+d-1} + \cdots + p_0(n)u_n, \quad p_i \in \mathbb{Q}[n]$

$u_0, u_1, \ldots, u_{d-1} \in \mathbb{Q}$

$\forall n \in \mathbb{N}, u_n \geq 0$?

Let m be the lcm of denominators of c_i and the initial conditions,

$u_n \to v_n = m^n u_n \in \mathbb{Z}$

$v_n^2 - 1 \geq 0 \implies \forall n \in \mathbb{N}, u_n \neq 0$

Skolem Problem:

Input: $c_d u_{n+d} = c_{d-1}u_{n+d-1} + \cdots + c_0 u_n, \quad c_i \in \mathbb{Q}$

$u_0, u_1, \ldots, u_{d-1} \in \mathbb{Q}$

$\exists n \in \mathbb{N}, u_n = 0$?

Decidable for $d \leq 4$

Motivations

Combinatorial inequalities

Gillis, Reznick and Zeilberger

$$u_n = \sum_{k=0}^{n} (-1)^k \frac{(4n-3k)!(4!)^k}{(n-k)!^4 k!} \geq 0,\ n \in \mathbb{N}$$

Special functions inequalities

Turán's inequality

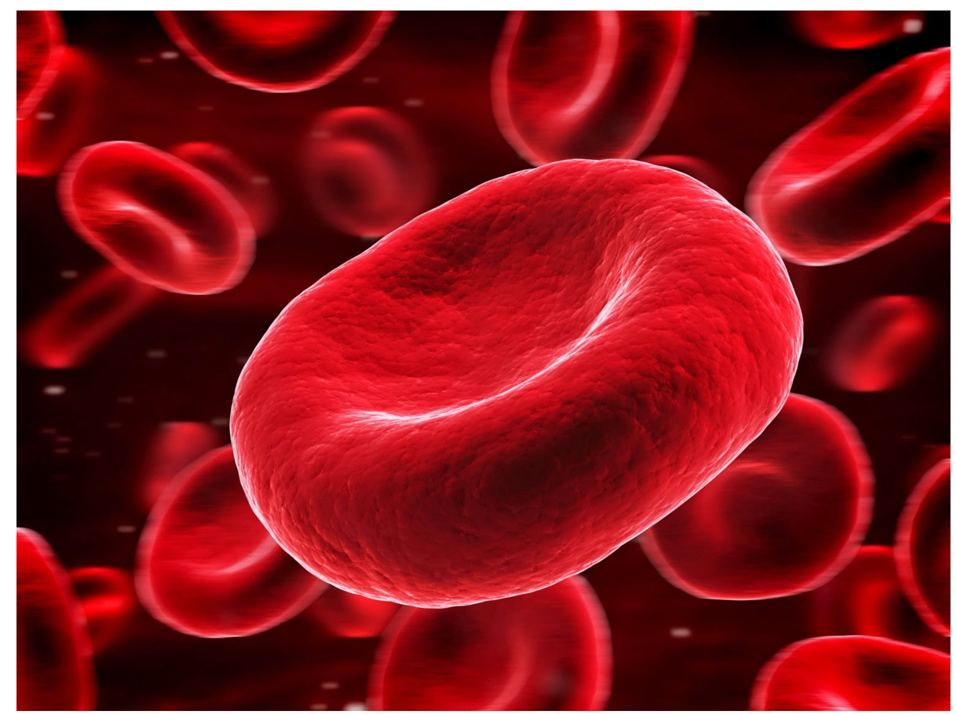
$$P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) \geq 0,\ x \in (0,1)$$

Numerical stability

Probability of collision between a satellite and orbiting debris

D. Arzelier, F. Bréhard, M. Masson, J.-B. Lasserre, B. Salvy, R. Serra 2023

Biology



Uniqueness of the Canham model for biomembranes

Melczer, Mezzarobba 2022

II. C- finite sequences

Closed form

$$c_d u_{n+d} = c_{d-1} u_{n+d-1} + \cdots + c_0 u_n,\ u_0, u_1, \ldots u_{d-1}$$ C-finite sequence

$$\chi_u(X) = c_d X^d - \sum_{i=0}^{d-1} c_i X^i = c_d \prod_{i=1}^{k} (X - \lambda_i)^{m_i}, \quad \lambda_i \in \mathbb{C}$$ Characteristic polynomial

$$u_n = \sum q_i(n) \lambda_i^n, \qquad q_i \in \mathbb{Q}[n],\ \deg(q_i) < m_i$$ Closed form

The q_i depend on the initial conditions

Special case: $\lambda_1 > |\lambda_2| \geq |\lambda_3| \ldots \geq |\lambda_k|, m_1 = 1$

How to check the positivity of $(u_n)_n$?

$$u_n = \mathbf{a}\lambda_1^n + \sum \lambda_i^n p_i(n)$$

$$\frac{u_n}{\lambda_1^n} = a + \mathcal{O}\left(\frac{\lambda_2^n}{\lambda_1^n}\right)$$

Decidability results

$$c_d u_{n+d} = c_{d-1} u_{n+d-1} + \cdots + c_0 u_n, \qquad \chi_u(X) = c_d \prod_{i=1}^{k} (X - \lambda_i)^{m_i}$$

[Ouaknine, Worrell et al.] For C-finite sequences positivity is decidable in these cases:

- $d < 5$
- $d \in \mathbb{N}, |\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_k|$ and the initial conditions are generic

Hardness:

Open problems in diophantine approximation → Positivity problem for $d = 6$

Reduction

III. Eigenvalues of a linear recurrence

Recurrence to matrix

$$p_d(n)u_{n+d} = p_{d-1}(n)u_{n+d-1} + \cdots + p_0(n)u_n, \qquad p_i \in \mathbb{Q}[n]$$

Let $U_n = (u_n, u_{n+1}, \ldots, u_{n+d-1})^t$, then

$$U_{n+1} = A(n)U_n = A(n)\cdots A(0)\, U_0 \quad A(n) = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & 1 \\ \frac{p_0(n)}{p_d(n)} & \frac{p_1(n)}{p_d(n)} & \frac{p_2(n)}{p_d(n)} & \ldots & \frac{p_{d-1}(n)}{p_d(n)} \end{pmatrix}$$

Assumption 1: $A := \lim_{n\to\infty} A(n) \in \mathrm{GL}_d(\mathbb{Q})$

The eigenvalues of the recurrence are the eigenvalues of the matrix A

Dominant eigenvalues

$\lambda_1, \ldots, \lambda_\nu$ are the **dominant eigenvalues** of A, or dominant roots of $\prod_{i=1}^{k}(X-\lambda_i)^{m_i}$
if $|\lambda_1| = |\lambda_2| = \cdots = |\lambda_\nu| > |\lambda_{\nu+1}| \geq |\lambda_{\nu+2}| \cdots \geq |\lambda_k|$

The power method:

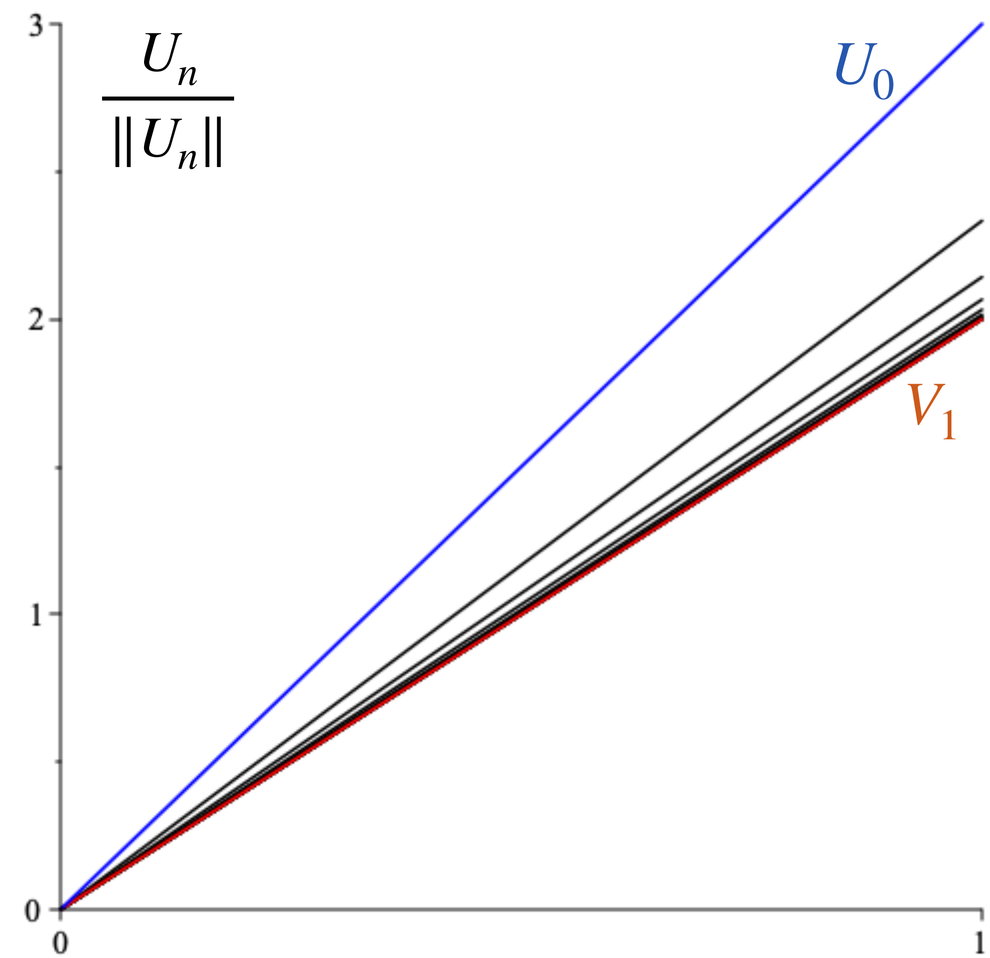
Assumption 2: $\lambda_1 > |\lambda_2| \geq |\lambda_3| \cdots \geq |\lambda_k|$ and λ_1 is simple ($m_1 = 1$)

$$AV_i = \lambda_i V_i,\ U_0 = a_1 V_1 + a_2 V_2 + \cdots,$$

U_0 **generic** ($a_1 \neq 0$)

$$U_n = A^n U_0 = a_1 \lambda_1^n V_1 + a_2 \lambda_2^n V_2 \cdots$$

$$\frac{U_n}{a_1 \lambda_1^n} = V_1 + \mathcal{O}\left(\frac{\lambda_2^n}{\lambda_1^n}\right)$$



IV. P-finite sequences

Leading coefficients in the closed form

The number of fragmented permutations of size n is $c_n/n!$,

$$(n+2)c_{n+2} = (2n+3)c_{n+1} - nc_n, \quad c_0 = c_1 = 1.$$

$$c_n \sim \frac{n^{-3/4}e^{2\sqrt{n}}}{2\sqrt{e\pi}}, \quad n \to \infty .$$

Previous works

$$p_d(n)u_{n+d} = p_{d-1}(n)u_{n+d-1} + \cdots + p_0(n)u_n, \qquad p_i \in \mathbb{Q}[n]$$

- [Gerhold-Kauers 2005 *GK*]: Find the first $k \in \mathbb{N}$ such that

$$(u_n \geq 0, \ldots, u_{n+k} \geq 0) \implies u_{n+k+1} = \sum_{i=0}^{d-1} r_i(n)u_{n+i} \geq 0$$

Such k does not necessarily exist

- [Kauers-Pillwein 2010]: By GK method find $\beta > 0$, $u_{n+1} > \beta u_n > 0$.

$$(u_{n+1} \geq \beta u_n, \ldots, u_{n+k} \geq \beta u_{n+k-1}) \implies u_{n+k+1} = \sum_{i=0}^{d-1} r_i(n)u_{n+i} \geq \beta u_{n+k}$$

Such k does not necessarily exist

- [Melczer, Mezzarobba and others]: Analytic methods

Decidability results:

$$p_d(n)u_{n+d} = p_{d-1}(n)u_{n+d-1} + \cdots + p_0(n)u_n, \qquad p_i \in \mathbb{Q}[n]$$

[Kauers-Pillwein 2010]: Under the following assumptions,

Assumption 1: $A := \lim_{n\to\infty} A(n) \in \mathrm{GL}_d(\mathbb{Q})$

Assumption 2: A has one simple dominant eigenvalue

Assumption 3: U_0 is generic

positivity is decidable for $d = 2,$ and **subclasses** of recurrences for $d = 3$

V. New result

Our result

$$p_d(n)u_{n+d} = p_{d-1}(n)u_{n+d-1} + \cdots + p_0(n)u_n, \qquad p_i \in \mathbb{Q}[n]$$

[I., Salvy 2024]: Under the following assumptions,

Assumption 1: $A := \lim_{n\to\infty} A(n) \in \mathrm{GL}_d(\mathbb{Q})$

Assumption 2: A has one simple dominant eigenvalue

Assumption 3: U_0 is generic

positivity is decidable for **arbitrary order** d**.**

Under these assumptions, we give an **algorithm** that proves positivity with **certificates**

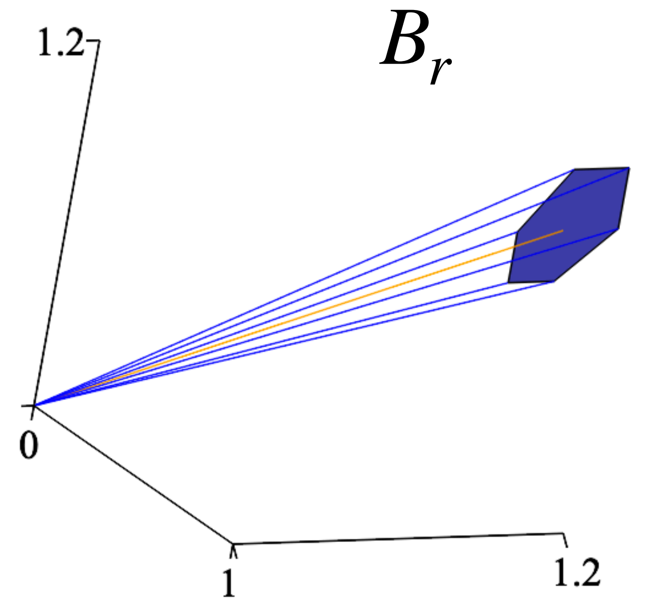
Certificate: Data-structure for a proof by induction

Input: $A(n) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{5n-7}{60(n+1)} & \frac{-(2n-1)}{3(n+1)} & \frac{19n+55}{12(n+1)} \end{pmatrix}, U_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

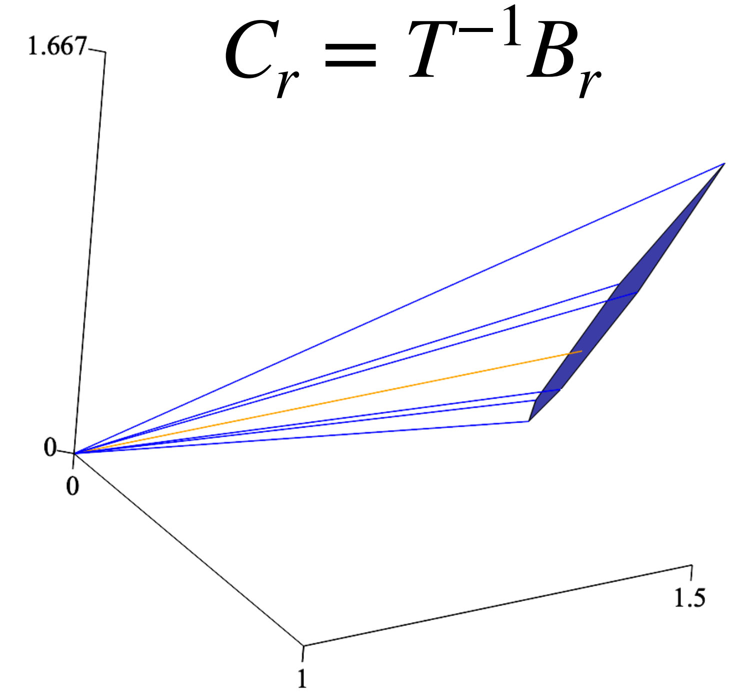
Output: Positive, $T = \begin{pmatrix} 1 & -4 & 4 \\ 0 & -2 & 3 \\ 0 & 0 & 1 \end{pmatrix}, r = 6/5, n_0 = 80.$

$$B_r := \{(x_1, x_2, x_3) \in \mathbb{R}^3_{>0} \mid x_i \leq r x_j\}$$

B_r



$C_r = T^{-1} B_r$



Verification steps

$$U_n = (u_n, u_{n+1}, u_{n+2}), U_{n+1} = A(n)U_n$$

1. Check by induction that $U_n \in C_r, n \geq n_0$

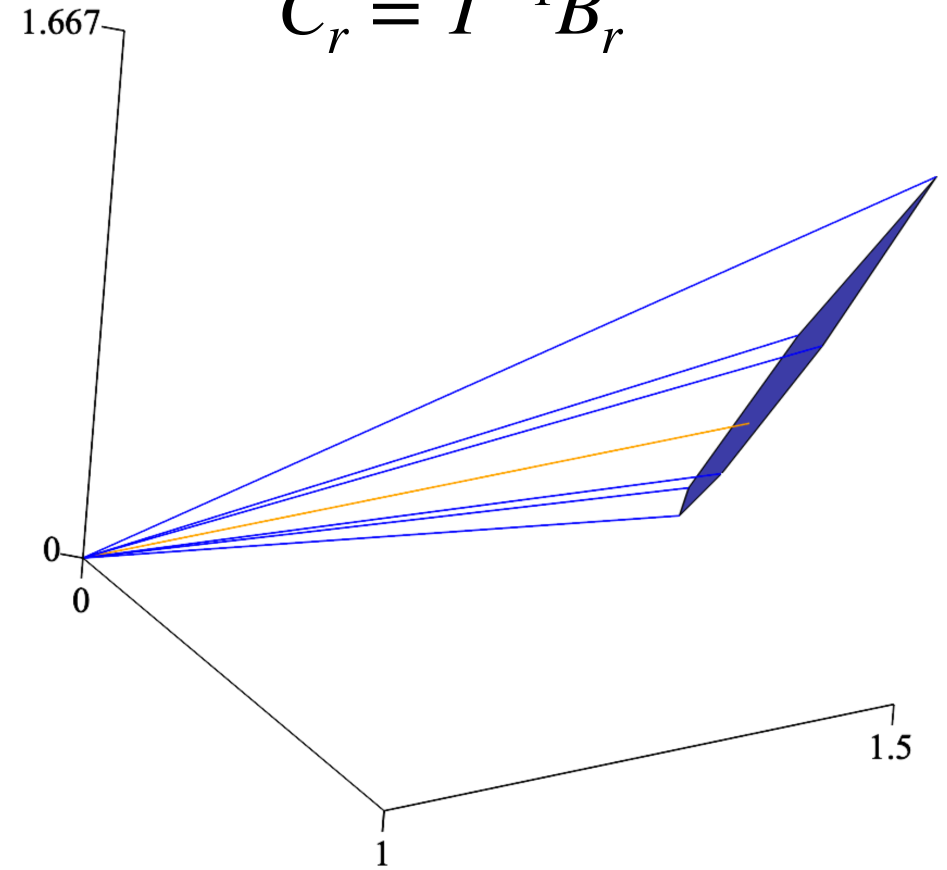
$$U_{n_0} \in C_r$$

$$\forall n \geq n_0,\ A(n)C_r \subset C_r$$

$$\implies \forall n \geq n_0, U_n \in C_r$$

$$\implies \forall n \geq n_0, u_n > 0$$

$$C_r = T^{-1}B_r$$



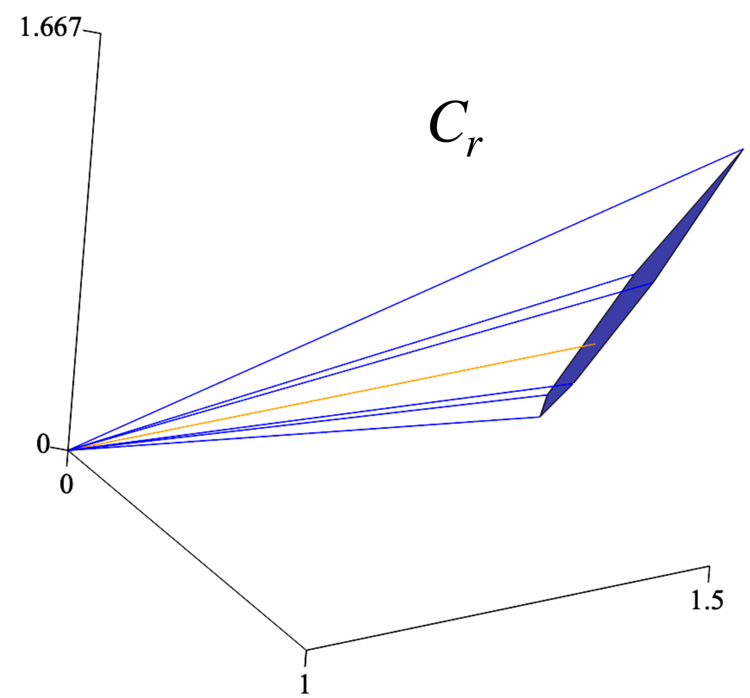
2.Check initial conditions:

$$u_n > 0,\ n < n_0$$

$$\implies \forall n \geq 0, u_n > 0$$

$$C_r = \left\{ y \in \mathbb{R}^3 \mid \frac{8y_2}{5} - \frac{2y_3}{5} > y_1, 4y_2 - \frac{14y_3}{5} > y_1, \frac{13y_3}{5} > \frac{12y_2}{5}, \frac{6y_1}{5} + \frac{9y_3}{5} > \frac{14y_2}{5}, 2y_2 > \frac{9y_3}{5}, \frac{6y_1}{5} + \frac{19y_3}{5} > \frac{24y_2}{5} \right\}$$

$A(n)C_r \subset C_r \Leftrightarrow A(n)V \in C_r$ for all extremal vector V of C_r



Example: For $V = \left(\frac{6}{5}, \frac{6}{5}, 1\right), A(n)V = \left(1, 1, \frac{305n + 1433}{300(n+1)}\right)$

$A(n)V \in C_r \Leftrightarrow$ All the following polynomials are positive

$\{145n - 983, 115n - 7781, 115n + 3499, 85n - 3299, 395n + 21827, 365n + 15029\}$

$\implies n \geq 68$

VI. Convergence of the product of matrices

Generalised power method

Theorem [Friedland 2004]:

$A_k \in \mathrm{GL}_d(\mathbb{R}), k \in \mathbb{N}$

$\{A_k\}_k \to A \neq 0$

A has one simple dominant eigenvalue λ_1,

Assumption 2:
$\lambda_1 > |\lambda_2| \geq |\lambda_3| \cdots \geq |\lambda_k|$
and λ_1 is simple

$$\implies \lim_{n\to\infty} \frac{A_n \cdots A_1 A_0}{\|A_n \cdots A_1 A_0\|} = V_1 w^t, \qquad w \in \mathbb{R}^d \text{ and } AV_1 = \lambda_1 V_1$$

$$U_{n+1} = A_n U_n = A_n \cdots A_0 U_0$$

Corollary $w^t U_0 \neq 0 \implies \lim_{n\to\infty} \frac{U_n}{\|U_n\|} = cV_1,\ c \neq 0.$

U_0 is said generic.

Generic initial conditions

Assumption 3: U_0 is generic

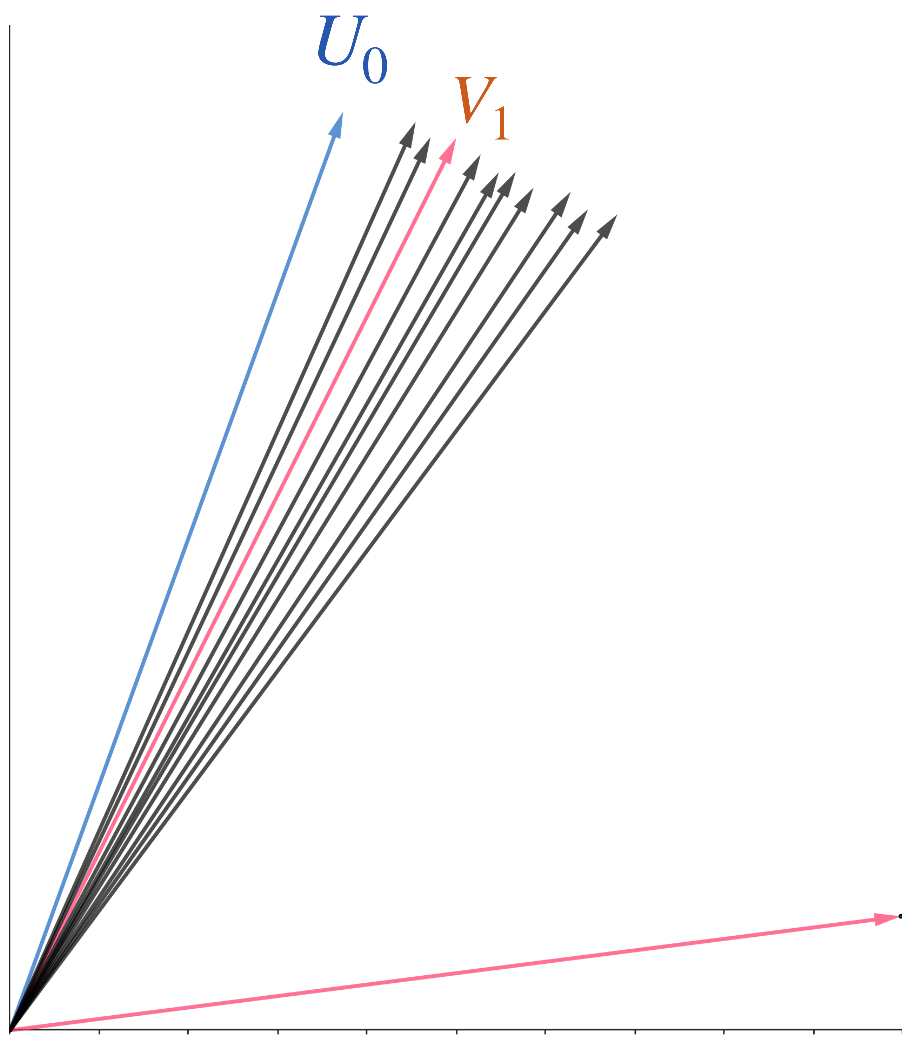
Apéry's recurrence:

$$\zeta(3) := \sum_{n=1}^{\infty} \frac{1}{n^3} \notin \mathbb{Q}$$

$$(n+2)^3 u_{n+2} = (2n+3)(17n^2+51n+39)u_{n+1} - (n+1)^3 u_n$$

$$\lambda_1 = 17 + 12\sqrt{2}$$

$$V_1 = (1, \lambda_1), \; w^t = (1, 6/\zeta(3) - 5).$$



Contraction of positive matrices

Theorem [Birkhoff 1957]

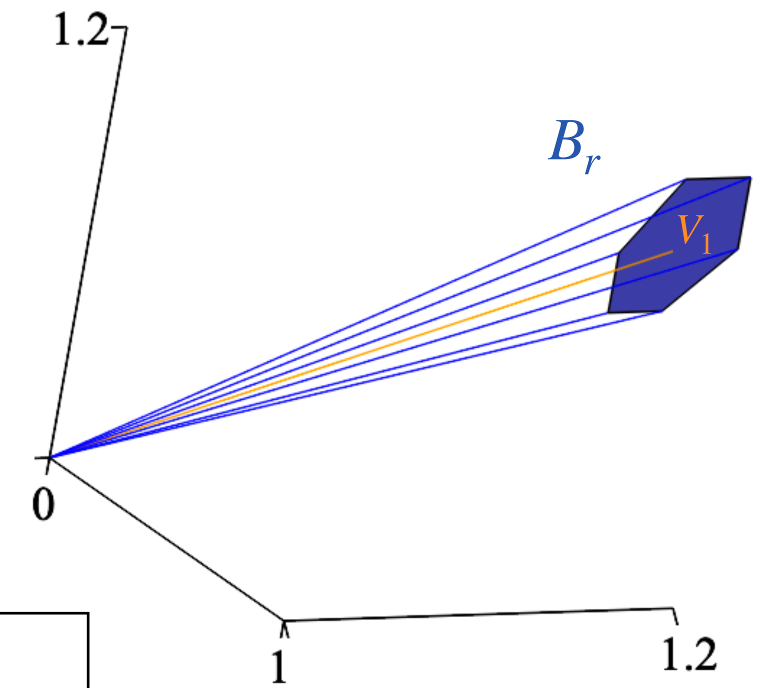
$A \in \mathbb{R}^{d\times d}, A > 0$
$\implies A$ is a contraction in the positive projective space $\mathbb{P}_{d-1}(\mathbb{R}_{>0})$, for the Hilbert metric d_H.

Perron-Frobenius: Let $A > 0$,

$\implies \exists V_1 > 0,\ \lambda_1 > 0 \,|\, AV_1 = \lambda_1 V_1$

Birkhoff

For $r > 1$, let $B_r = B_{d_H}(V_1, \log r)$ then $AB_r \subset \mathring{B}_r$



Assume $V_1 = (1,1,\ldots,1)$,
$B_r = \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d_{>0},\ x_i \leq r x_j\}$
Extremal vectors:
$\{(v_1, \ldots, v_d), v_i \in \{1,r\}\}\backslash\{(1,\ldots,1), (r, \ldots, r)\}$

Effective Friedland theorem

Assumption 2: A has one simple dominant eigenvalue λ_1. $AV_1 = \lambda_1 V_1$

↓ Friedland's lemma

$\exists T \in GL_d(\mathbb{Q})$ such that $TAT^{-1} > 0$

↓ Birkhoff's theorem

Choose $r \mid T^{-1}B_r \subset \mathbb{R}^d_{>0}$. $TAT^{-1}B_r \subset \mathring{B}_r$

↓ $A(n) \to A$

$\exists m \in \mathbb{N} \mid TA(n)T^{-1}B_r \subset B_r , \forall n \geq m$

↓

$\exists m \in \mathbb{N} \mid A(n)T^{-1}B_r \subset T^{-1}B_r , \forall n \geq m$

↓ $U_{n+1} = A(n)U_n$

$\forall n \geq m, U_n \in T^{-1}B_r \implies U_{n+1} \subset T^{-1}B_r$

↓ Generically

$\lim_{n\to\infty} \frac{U_n}{\|U_n\|} = V_1$

↓

$\exists n_0 \geq m \mid U_{n_0} \subset T^{-1}B_r$

↓

$\forall n \geq n_0, U_n \subset T^{-1}B_r \subset \mathbb{R}^d_{>0}$
$\implies \forall n \geq n_0, U_n \geq 0$

Algorithm 1

Soda'24

Input: $A(n) \in (\mathbb{Q}[n])^{d\times d}$,

$U_0 \in \mathbb{Q}^d$

Assumption 1: $A := \lim_{n\to\infty} A(n) \in \mathrm{GL}_d(\mathbb{Q})$

Assumption 2: A has one simple dominant eigenvalue λ_1

Output: Positive (T, r, n_0)/ **Non Positive**

1. Construct $C_r = T^{-1}B_r \in \mathbb{R}^d_{>0}$ by computing $T \in \mathrm{GL}_d(\mathbb{Q})$ and $r > 1$ ($V_1 \in C_r$).
   Linear system in one variable ($2^d - 2$ inequalities)

2. Find m s.t. for all $n \geq m$, $A(n)C_r \subset C_r$
   $d(d-1)$ inequalities $\in \mathbb{Q}[n]$ for each border vector ($2^d - 2$ vectors)

3. Find $n_0 \geq m$ s.t. $U_{n_0} \in C_r$ and $U_n \geq 0$ for $n \leq n_0$

Examples

- Straub, Zudilin 2014: $d = 2$

$$2(n+2)^2 s_{n+2} = 3(27n^2 + 81n + 62)s_{n+1} - 81(3n+2)(3n+4)s_n, \quad s_0 = 1,\ s_1 = 12$$

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> PositivityProof(rec,s,n,27);
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$$true, \left[T = \begin{bmatrix} -28 & \frac{29}{27} \\ 26 & -\frac{25}{27} \end{bmatrix}, m = 1, r = \infty, n0 = 30 \right]$$

$d = 3$

- $(n+1)u_{n+3} = (77n/30 + 2)u_{n+2} + (-13n/6 + 3)u_{n+1} + (3n/5 + 2)u_n,\ u_0 = 1,\ u_1 = 15/14,\ u_2 = 8/7$

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> PositivityProof(rec,u,n,1);
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$$true, \left[T = \begin{bmatrix} 54 & -142 & 89 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, m = 30, r = \frac{98}{61}, n0 = 11209 \right]$$

How to reduce n_0?

VII. Matrices leaving a cone invariant

Jordan form

$$A \sim \begin{bmatrix} \lambda_1 & 1 & & & & & & & & & \\ & \lambda_1 & 1 & & & & & & & & \\ & & \lambda_1 & & & & & & & & \\ & & & \lambda_1 & 1 & & & & & & \\ & & & & \lambda_1 & & & & & & \\ & & & & & \lambda_2 & 1 & & & & \\ & & & & & & \lambda_2 & & & & \\ & & & & & & & \lambda_2 & & & \\ & & & & & & & & \lambda_3 & 1 \\ & & & & & & & & & \lambda_3 \end{bmatrix}$$

$m(\lambda_i)$ = maximal dimension of the Jordan blocks of A containing λ_i

Example: $m(\lambda_1) = 3$

Perron-Schaefer condition

$A \in \mathbb{R}^{d \times d}$

$$\chi_A(X) = \prod_{i=1}^{k} (X - \lambda_i)^{m_i}, \quad \lambda_i \in \mathbb{C}$$

$\rho(A) := \max |\lambda_i|$

Theorem [Birkhoff 1967, Vandergraft 1968]

There exists a cone K such that $\mathring{K} \neq \emptyset$ and $AK \subset K$ if and only if

1. $\rho(A)$ is an eigenvalue of A

2. If $|\lambda_i| = \rho(A)$ then $m(\lambda_i) \leq m(\rho(A))$

$A(n) \to A$ and $AK \subset K \not\Longrightarrow \exists n_0 \,|\, A(n)K \subset K,\ n > n_0$

Contraction

$$A \in \mathbb{R}^{d\times d}$$

$$\chi_A(X) = \prod_{i=1}^{k} (X - \lambda_i)^{m_i}, \quad \lambda_i \in \mathbb{C}$$

$$\rho(A) := \max |\lambda_i|$$

Theorem [Vandergraft 1968]

There exists a solid cone K such that $\mathring{K} \neq \varnothing$ and $AK \subset \mathring{K}$ if and only if

1. $\rho(A)$ is a **simple** eigenvalue of A
2. $\rho(A) > |\lambda_i|$, for all $\lambda_i \neq \rho(A)$

$$A(n) \to A \text{ and } AK \subset \mathring{K} \implies \exists n_0 | A(n)K \subset K, \forall n > n_0$$

Sketch of the proof:

Assumption: All the eigenvalues $(\lambda_i)_{i=1}^{d}$ of A are simple and real

Conditions 1 and 2 $\Longrightarrow \lambda_1 = \rho(A)$ and $\lambda_1 > |\lambda_i|$ for $i \neq 1$

For $i = 1, \ldots, d$, let $AV_i = \lambda_i V_i$

Let $K := \{a_1 V_1 + \sum_{i=2}^{d} a_i V_i,\ |a_i| \leq a_1\}$,

$$AK = \underbrace{a_1 \lambda_1}V_1 + \sum_{i=2}^{d} \underbrace{a_i \lambda_i} V_i \quad \subset \mathring{K}$$

Illustration of the algorithm

$$2(n+2)^2 s_{n+2} = 3(27n^2+81n+62)s_{n+1} - 81(3n+2)(3n+4)s_n, \quad s_0 = 1,\ s_1 = 12$$

$$A(n) = \begin{pmatrix} 0 & 1 \\ \frac{-81(3n+2)(3n+4)}{2(n+2)^2} & \frac{3(27n^2+81n+62)}{2(n+2)^2} \end{pmatrix} \quad A = \begin{pmatrix} 0 & 1 \\ -729/2 & 81/2 \end{pmatrix}$$

1. Cone construction $AK \subset \mathring{K}$:

Eigenvalues: $\chi(A) = (X - \underbrace{27}_{\lambda_1})(X - \underbrace{27/2}_{\lambda_2})$

Basis of eigenvectors: $V_1 = (1,27),\ V_2 = (1,27/2)$

$AV_i = \lambda_i V_i$

A cone: $K = \{a_1V_1 + a_2V_2, a_1 \geq |a_2|\}$

Extremal vectors of K: $V_1 + V_2,\ V_1 - V_2$

$$K = \{a_1V_1 + a_2V_2, a_1 \geq |a_2|\} = \{b_1(V_1 + V_2) + b_2(V_1 - V_2), b_i \geq 0\}$$

$$A(n)K \subset K \Leftrightarrow A(n)(V_1 \pm V_2) \subset K$$

2. Find $m \in \mathbb{N}$ s.t $\forall\, n \geq m, A(n)K \subset K$:

$$A(n)(V_1 + V_2) = \underbrace{\frac{3(18n^2 + 63n + 46)}{2(n^2 + 4n + 4)}}_{a_1} V_1 + \underbrace{\frac{3(9n^2 + 45n + 62)}{2(n^2 + 4n + 4)}}_{a_2} V_2$$

$\Longrightarrow$ Solve a system of inequalities in **one** variable:

$\{n > 0, (18n^2 + 63n + 46) \geq 3(9n^2 + 45n + 62)\} = \{n > 0\}$

Repeat the same steps for $A(n)(V_1 - V_2) \in K$

$\Longrightarrow m = 0$

3. Test initial conditions:

- Find the first $n_0 \geq m$ s.t $U_{n_0} \in K \quad (\forall n \geq n_0,\ U_n \in K \subset \mathbb{R}^2_{\geq 0})$

$n_0 = 3$

- Check the positivity of the n_0 first vectors U_n

Output: Positive, (K, n_0)

Algorithm 2

Input: $A(n) \in (\mathbb{Q}[n])^{d\times d},\ U_0 \in \mathbb{Q}^d$
$A(n) \rightarrow A \in \mathbb{Q}^{d\times d}$,

Assumption 2: A has one simple dominant eigenvalue λ_1

Output: Positive $(K,\ n_0)$/ Non Positive

1. Construct a cone K such that $AK \in \mathring{K}$

   Vandergraft construction

2. Find $m \in \mathbb{N}$ s.t $\forall\, n \geq m, A(n)K \subset K$

3. Initial conditions test:

   Find the first $n_0 \geq m$ s.t $U_{n_0} \in K$

   Check if $U_n > 0$ for $n < n_0$

Symbolic and
numerical methods
(Interval analysis)

Comparaison

- Straub, Zudilin 2014:

$$2(n+2)^2 s_{n+2} = 3(27n^2+81n+62)s_{n+1} - 81(3n+2)(3n+4)s_n, \quad s_0 = 1,\ s_1 = 12$$

Algorithme 1: $n_0 = 30$

Algorithme 2: $n_0 = 4$

- $(n+1)u_{n+3} = (77n/30+2)u_{n+2} + (-13n/6+3)u_{n+1} + (3n/5+2)u_n,\ u_0 = 1,\ u_1 = 15/14,\ u_2 = 8/7$

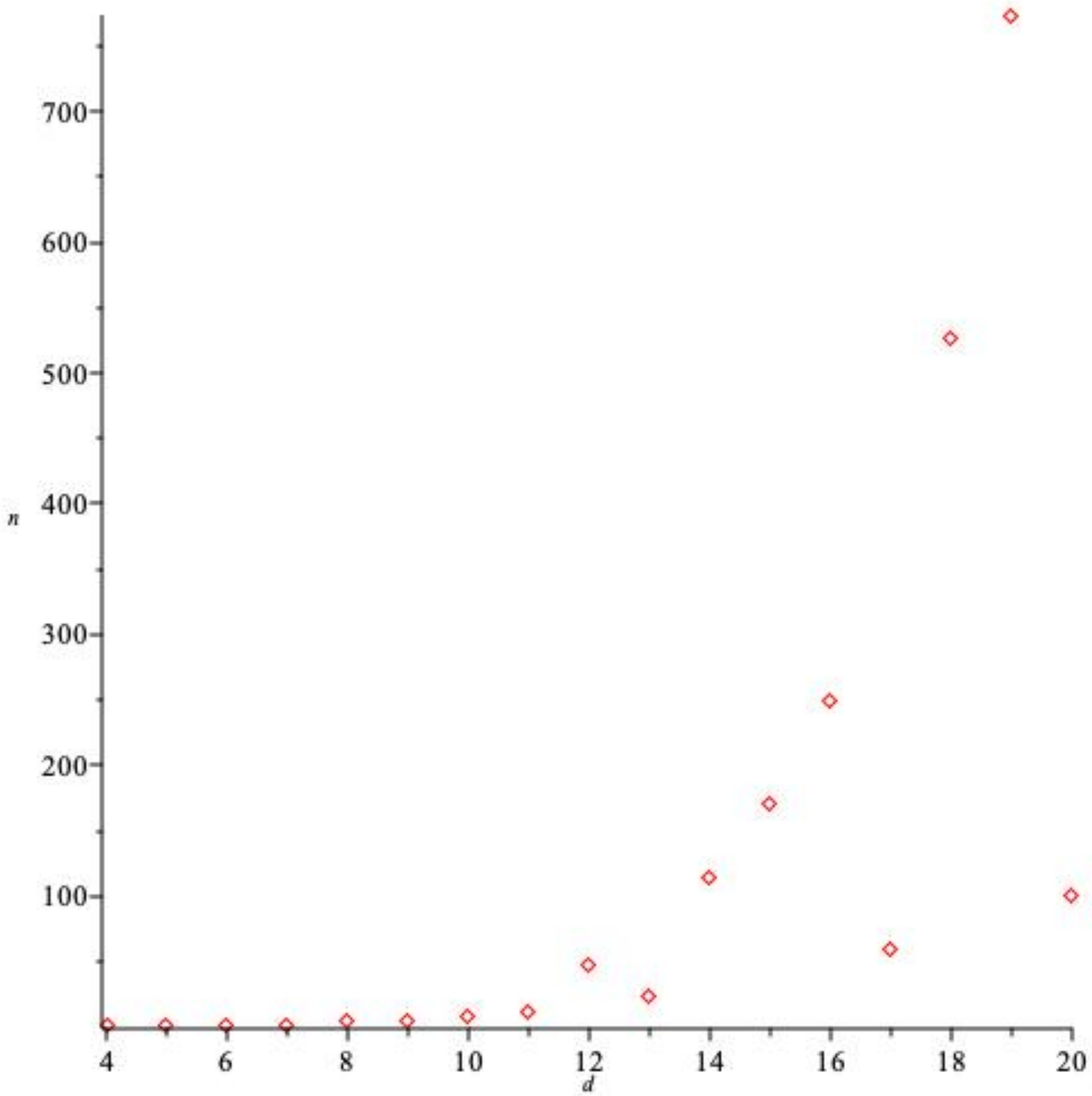
Algorithme 1: $n_0 = 11209$

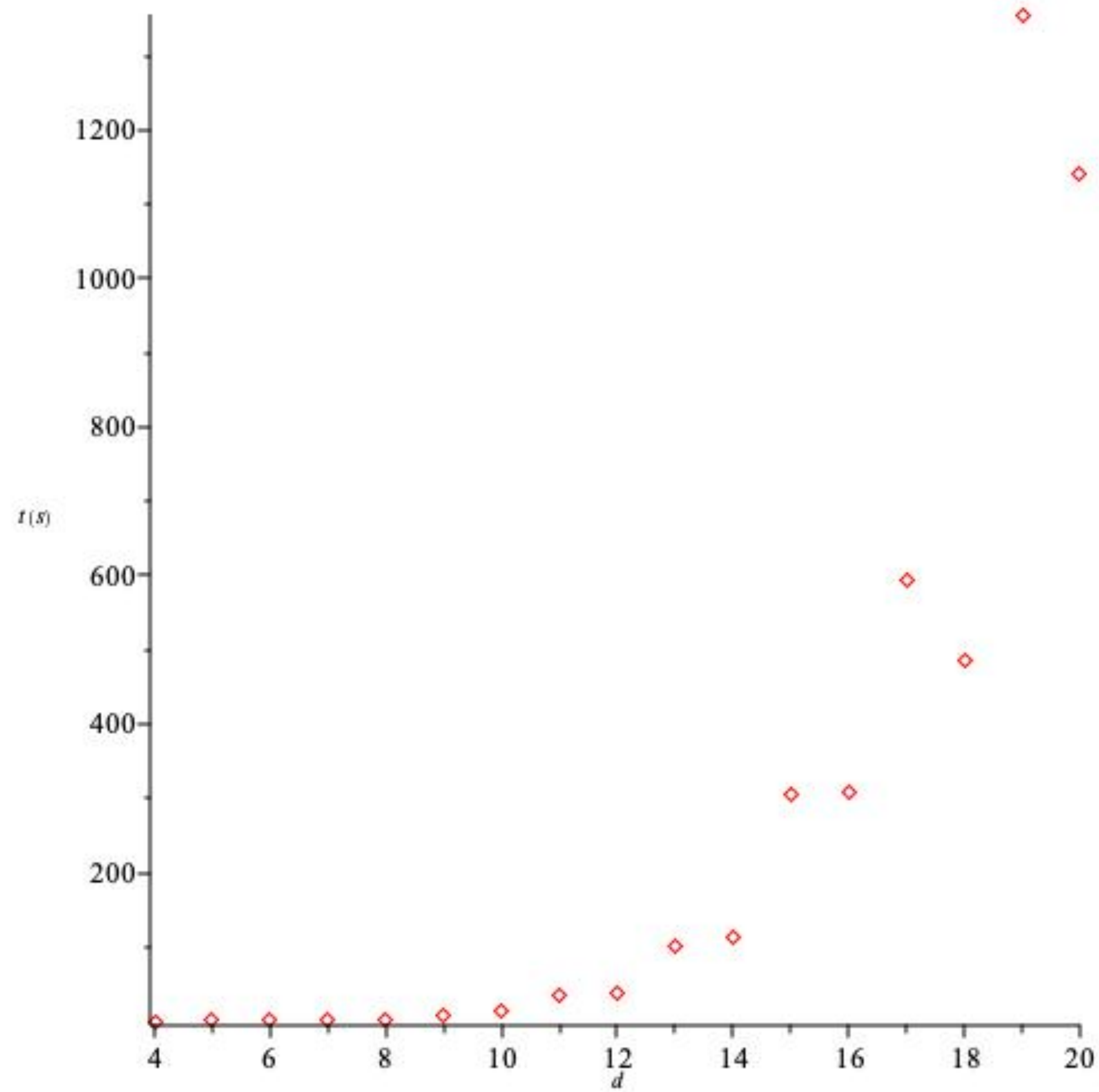
Algorithme 2: $n_0 = 1200$

- [Gillis, Reznick, Zeilberger 83] For $r = 4$, $d_{n,n,n,n} \geq 0$

$$\frac{1}{1 - \sum_{i=1}^{r} t_i + r! \prod_{i=1}^{r} t_i} = \sum \mathbf{d}_{\mathbf{i_1,i_2,\ldots,i_r}} \, t_1^{i_1} \ldots t_r^{i_r}$$

Algorithme 1: –





Conclusion

Positivity is decidable with **certificates** for all P-finite sequences with **one simple dominant** eigenvalue and **generic** initial conditions.

In progress:

- Sequences with parameters
- Sequences with several dominant eigenvalues which are simple